Angular momentum

Once the ideas of force and time had been introduced, it was natural to add the concept which meant the product of a force applied over a time span. This was linear impulse:

$\mathbf{J} = \mathbf{F}\cdot\Delta t$

It was then shown that this impulse was equal to a change in a quantity called linear momentum:

$\mathbf{J} = \Delta\mathbf{p}$ where $\mathbf{p} = m\cdot\mathbf{v}$

Likewise, for rotation, we can easily imagine applying a torque applied to some rotating system over a span of time and we can call this angular impulse:

$\mathbf{J}_{angular} = \boldsymbol{\tau}\cdot\Delta t$

Answer Webassign Question 1

What quantity will this angular impulse cause a change in?

We know $\boldsymbol{\tau} = I\cdot\boldsymbol{\alpha}$, so $\boldsymbol{\tau}\cdot\Delta t = I\cdot\boldsymbol{\alpha}\cdot\Delta t = I\cdot\boldsymbol{\Delta\omega}$

If we define angular momentum as rotational inertia times angular velocity, $\mathbf{L} = I\cdot\boldsymbol{\omega}$

Then $\mathbf{J}_{angular} = \boldsymbol{\Delta}\mathbf{L}$ or angular impulse equals a change in angular momentum

Answer Webassign Question 2

$\mathbf{L} = I\cdot\boldsymbol{\omega}$ is a useful equation for disks and spheres that spin around themselves, but there is another useful equation for angular momentum of point particles:

$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F}$

$\boldsymbol{\tau}\cdot\Delta t = \mathbf{R} \times \mathbf{F}\cdot\Delta t$

$\mathbf{J}_{angular} = \mathbf{R} \times (m\cdot\boldsymbol{\Delta}\mathbf{v})$

$\mathbf{J}_{angular} = \boldsymbol{\Delta}\mathbf{L}$ where $\mathbf{L}_{point\ mass} = \mathbf{R} \times \mathbf{p} = \mathbf{R} \times m\cdot\mathbf{v}$

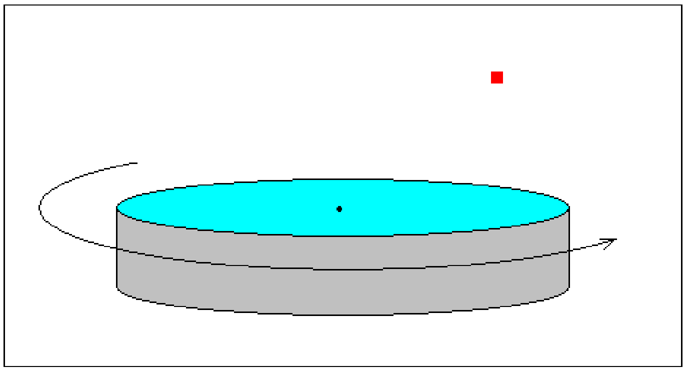
The magnitude can also be conceptualized as the product of the momentum ($m\cdot\mathbf{v}$) and the radius perpendicular to that momentum ($R_\perp$), so

$$|L| = R_\perp\cdot mv$$

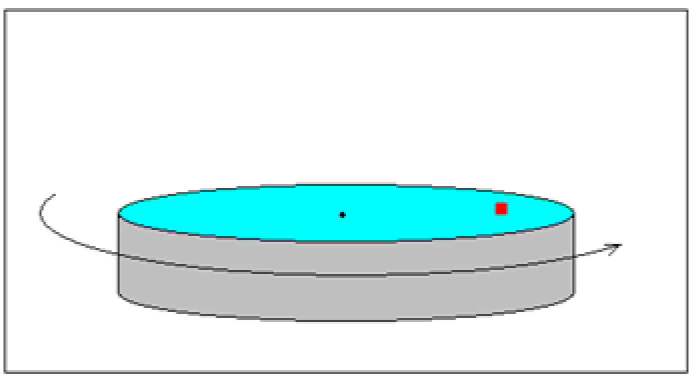
Answer Webassign Question 3

From the concept of linear momentum, we used Newton's third law of motion to derive conservation of linear momentum. We can do something similar with angular momentum to see that it too is conserved.

Suppose you have a horizontal disk spinning freely without friction:



Now you drop the little red clay cube onto the disk and it sticks.



Now the clay cube is spinning with the disk, but in the very brief process of the collision, something happened: the clay cube went from zero horizontal motion to spinning around. This means that the disk applied a torque to it, causing the clay cube to have an angular acceleration.

The torque of disk on cube implies a force of disk on cube. By Newton's third law, there is then a force of cube on disk, equal in magnitude but opposite in direction. Because this interaction happens at the same radius from the axis of rotation for both the disk and cube, the two objects experience equal and opposite torques, one speeding up the clay, another slowing down the disk.

The interaction lasts the same time for both, so both experience the same angular impulse, but in opposite directions. Therefore, for this closed system, $\Sigma \mathbf{J}_{angular} = 0$ and there is no overall change in angular momentum.

The condition that must be met is that the net external torque must be zero. This derivation works even if the force comes from outside the spinning disk, which I will leave for an appendix.

Let's see how this works with some numbers. Suppose the disk has a mass of 2kg, a radius of 20cm, and begins spinning at 10rad/s. The small red clay cube has a mass of 20g and is dropped straight down, landing 15cm from the axis of rotation.

Relative to the axis of rotation, the clay has zero angular momentum and the disk has:

$\mathbf{L} = I\cdot\boldsymbol{\omega} = (\frac{1}{2}M\cdot R^2)\cdot\omega = 0.4\ kg\cdot m^2\cdot\frac{rad}{s}$

The direction of this angular momentum vector can be determined from the right hand rule for rotation. Look at the diagram of the disk spinning and curl the fingers of your right hand with the rotation of the disk. Your right hand thumb naturally points upwards, so this is the direction of the angular momentum vector.

So the total initial angular momentum is 0 + 0.4 = 0.4, and because there are no external torques acting on the system, this value must be conserved.

When the clay has stuck to the wheel, the new rotational inertia of the spinning system will simply be the rotational inertia of the disk plus the rotational inertial of the point mass clay cube:

$I = (\frac{1}{2}M_{disk}\cdot R_{disk}^2) + (m_{clay}\cdot R_{clay}^2) = 0.04045\ kg\cdot m^2$

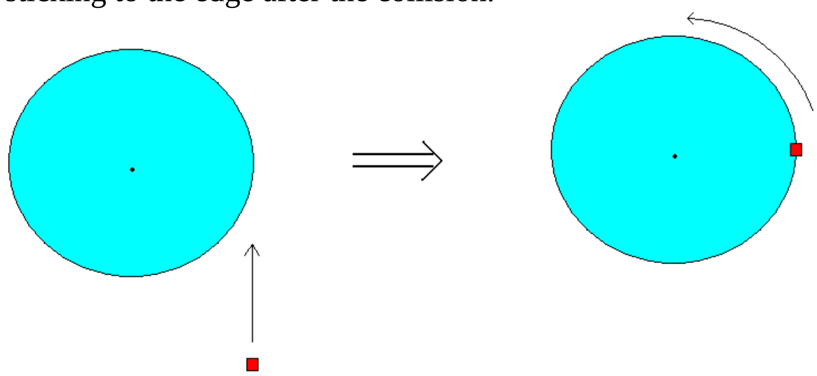
So $\mathbf{L}_{final} = I_{final}\cdot\omega_{final}$

$0.4\ kg\cdot m^2\cdot\frac{rad}{s} = (0.04045\ kg\cdot m^2)\cdot\ \omega_{final}$

$\omega_{final} = 9.89\ \frac{rad}{s}$ just slightly less than the initial angular velocity

Answer Webassign Question 4

Suppose instead, the clay cube was fired horizontally at 50m/s and struck the disk which begins at rest, sticking to the edge after the collision.



Here the initial angular momentum is in the clay alone:

$\mathbf{L} = \mathbf{R} \times m \cdot \mathbf{v} = |R| \cdot |mv| \cdot \sin\theta$

Again, the angle here is the angle between the first vector and the second, counter clockwise. When the clay hits, the radial vector is to the right and the velocity vector is upwards, so the angle is 90º.

$\mathbf{L} = (0.20\text{m})(0.02\text{kg})(50\text{m/s})(\sin 90^\circ) = 0.2\text{kg}\cdot\frac{m^2}{s}$

Angular momentum is conserved, so this is what we must end with as well.

The final rotational inertia is:

$I = (\frac{1}{2}M_{disk} \cdot R_{disk}^2) + (m_{clay} \cdot R_{clay}^2) = 0.0408 \text{ kg}\cdot\text{m}^2$

$\mathbf{L}_{final} = I_{final} \cdot \omega_{final}$

$0.2\text{kg}\cdot\frac{m^2}{s} = (0.0408 \text{ kg}\cdot\text{m}^2)(\omega_{final})$

$\omega_{final} = 4.9\frac{rad}{s}$

Answer Webassign Question 5

To review:

Angular impulse: $\mathbf{J} = \boldsymbol{\tau} \cdot \Delta t$ or

$\mathbf{J} = \boldsymbol{\Delta}\mathbf{L}$ where

$\mathbf{L} = I \cdot \boldsymbol{\omega}$ for spinning objects

$\mathbf{L} = \mathbf{R} \times \mathbf{p} = \mathbf{R} \times m \cdot \mathbf{v}$ for point masses

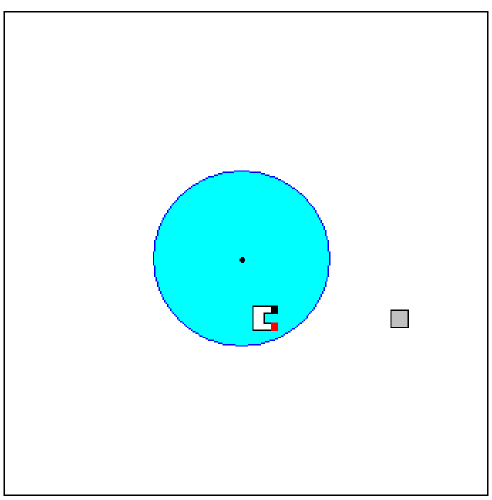
Rearranged becomes: $\boldsymbol{\tau} = \frac{\Delta L}{\Delta t}$

Angular momentum is conserved:

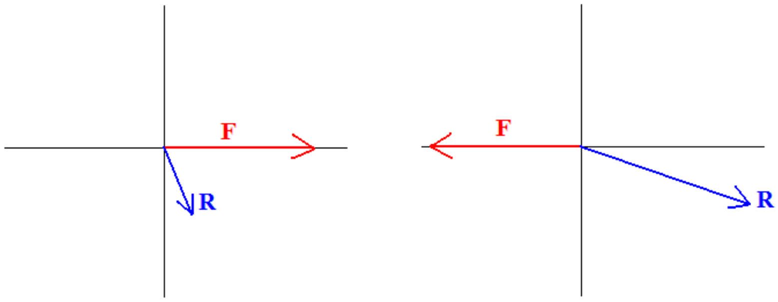
$\Sigma \mathbf{L}_{\mathbf{i}} = \Sigma \mathbf{L}_{\mathbf{f}}$ if $\Sigma \boldsymbol{\tau}_{\mathbf{ext}} = 0$

Appendix:

As seen from above, a disk is free to spin on a frictionless surface around a fixed point. Attached to the surface of the disk is a magnet. In the vicinity is a small steel cube which is free to slide on the same frictionless surface.



There is going to be an attraction between the magnet and steel cube, pulling them towards each other, making the cube move left and the disk spin counter-clockwise. By Newton's third law, the forces must be the same magnitude, but in opposite directions.



Magnet **Steel cube**

From the diagrams, you can see R_y for each is the same. The different R_x values will turn out to be irrelevant. If we then put each of these into a matrix for torque:

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R_x & R_y & R_z \\ F_x & F_y & F_z \end{matrix}$$

For the magnet, we have:

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R_x & R_y & 0 \\ F & 0 & 0 \end{matrix}$$

$\tau = -(F \cdot R_y)\hat{k}$

For the steel cube, we have:

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R_x & R_y & 0 \\ -F & 0 & 0 \end{matrix}$$

$\tau = (F \cdot R_y)\hat{k}$

The two are equal, but opposite and add to zero.