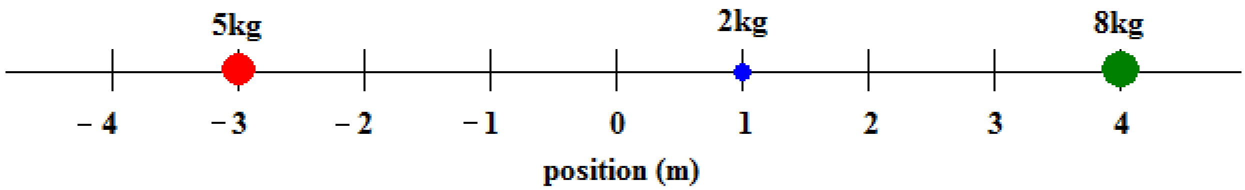
Center of mass

Conceptually, the center of mass of a system is the weighted average of the components' positions.

Let me illustrate that with an example.



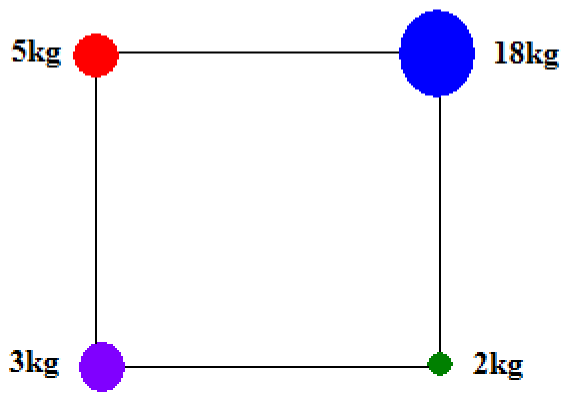
Above is a simple system of three masses. Where is the center of mass of the system? Well, we need to weight each mass with its position. The equation looks like this:

$$R_{cm} = \frac{m_1 \cdot R_1 + m_2 \cdot R_2 + m_3 \cdot R_3}{M} = \frac{(5kg)(-3m)+(2kg)(1m)+(8kg)(4m)}{15kg}$$

M is the total mass of the system, so here the center of mass, $R_{cm} = 1.2\bar{6}$m

Answer Webassign Question 1

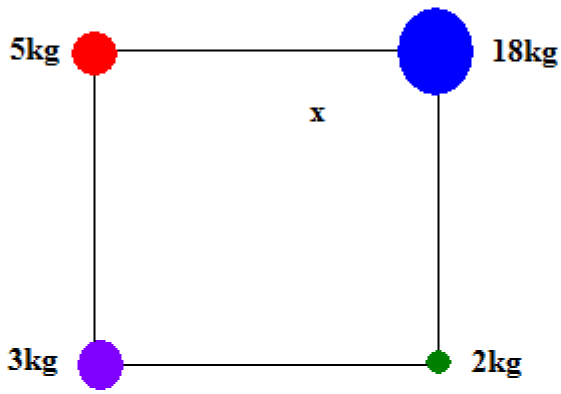
This concept can easily be extended into two and three dimensions. Suppose you had a square with sides of 2m. Relative to the bottom left corner, where is the center of mass?



In the x-dimension: $R_{cm} = \frac{(3kg)(0m)+(5kg)(0m)+(18kg)(2m)+(2kg)(2m)}{28kg} = 1.43\text{m}$

In the y-dimension, $R_{cm} = \frac{(3kg)(0m)+(5kg)(2m)+(18kg)(2m)+(2kg)(0m)}{28kg} = 1.64\text{m}$

So R_{cm} = < 1.43, 1.64 >m. That puts the center of mass about where the little *x* is below:



Incidentally, the center of mass is the point on which a system would balance (i.e. if you take a tennis racket and balance it on a fingertip, the racket's center of mass is directly above that fingertip). This is because both torque ($\tau = R \times F$) and center of mass are linear functions of position. It's also the point around which the system naturally spins (if you flip the racket up into the air, spinning around itself) because centripetal force is also a linear function of position ($F_C = m \cdot \omega^2 \cdot R$). Torque ($\tau$) and angular velocity ($\omega$) are concepts we'll see later.

Answer Webassign Question 2

This concept of center of mass can be related to velocity and momentum, acceleration and net force. Begin with basic equation for center of mass:

$$R_{cm} = \frac{m_1 \cdot R_1 + m_2 \cdot R_2 + m_3 \cdot R_3}{M}$$

If the masses move around, the center of mass can change as well, so there can be an initial and final center of mass, as well as a change in the center of mass:

$$R_{cm,i} = \frac{m_1 \cdot R_{1,i} + m_2 \cdot R_{2,i} + m_3 \cdot R_{3},i}{M} \qquad R_{cm,f} = \frac{m_1 \cdot R_{1,f} + m_2 \cdot R_{2,f} + m_3 \cdot R_{3},f}{M}$$

$$\Delta R_{cm} = R_{cm,f} - R_{cm,i} = \frac{m_1 \cdot \Delta R_1 + m_2 \cdot \Delta R_2 + m_3 \cdot \Delta R_3}{M}$$

and then dividing both sides by a time, Δt

$$v_{cm} = \frac{m_1 \cdot v_1 + m_2 \cdot v_2 + m_3 \cdot v_3}{M}$$

Answer Webassign Question 3

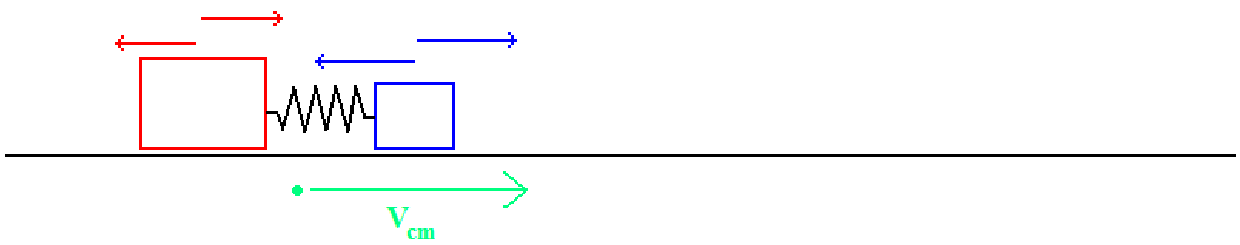
Repeating this process produces:

$$a_{cm} = \frac{m_1 \cdot a_1 + m_2 \cdot a_2 + m_3 \cdot a_3}{M}$$

Now, $m_1 \cdot a_1$ is the net force on particle one, $m_2 \cdot a_2$ is the net force on particle two, etc. All internal forces within the system will add to zero by Newton's third law, so the right hand numerator of the above equation becomes the sum of all external forces on the system, thus:

$\Sigma F_{ext} = (M)(a_{cm})$ where, again, M is the mass of the system

For example, take two blocks connected by a spring and the system sliding to the right. The blocks themselves may oscillate back and forth relative to the center of mass, but the center of mass itself will simply glide to the right with zero acceleration because the net force on the system is zero. The two spring forces are internal forces. Here's a video that shows that. It can help to see how the center of mass moves smoothly by covering-up the two gliders with your hands.



Or suppose you throw a tennis racket across a football field. The racket is likely to wobble around itself in a very complex pattern with many internal forces. But the total external force on the racket is just $-M \cdot g$,

so $-M \cdot g = (M)(a_{cm})$ and $a_{cm} = -g$.

The center of mass of the racket is going to move with a nice, simple acceleration of $-g$, in a parabolic path, just like a ball tossed across the field. Here's a video that shows that.

Answer Webassign Question 4

We can also take $\Sigma F_{ext} = (M)(a_{cm})$ and multiply both sides by Δt. Then

$$(\Sigma F_{ext})(\Delta t) = (M)(a_{cm})(\Delta t)$$

$$\Sigma J_{ext} = (M)(\Delta v_{cm}) = m_1 \cdot \Delta v_1 + m_2 \cdot \Delta v_2 + m_3 \cdot \Delta v_3$$

$$\Sigma J_{ext} = \Delta m_1 v_1 + \Delta m_2 v_2 + \Delta m_3 v_3$$

which reads: the total impulse imparted to a system equals the net change in momentum of that system.

Answer Webassign Question 5

To review:

$$R_{cm} = \frac{m_1 \cdot R_1 + m_2 \cdot R_2 + m_3 \cdot R_3 + \cdots}{M}$$

$$v_{cm} = \frac{m_1 \cdot v_1 + m_2 \cdot v_2 + m_3 \cdot v_3 + \cdots}{M}$$

$$a_{cm} = \frac{m_1 \cdot a_1 + m_2 \cdot a_2 + m_3 \cdot a_3 + \cdots}{M}$$

$$\Sigma F_{ext} = (M)(a_{cm})$$

A net external force on a system will cause the center of mass of that system to accelerate.

$$\Sigma J_{ext} = \Delta m_1 v_1 + \Delta m_2 v_2 + \Delta m_3 v_3 + \cdots$$

The total impulse imparted to a system will cause an equivalent change in momentum of that system.