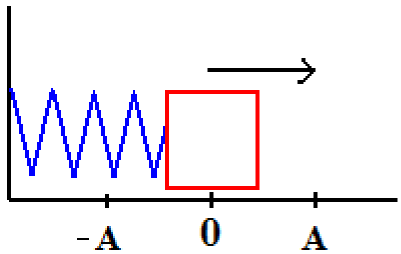
Mass and spring oscillators



On a frictionless surface, you have a block with mass *m* attached to a spring with a spring constant *k*. You pull the block to the right to position *A* and release it.

From there, it will oscillate between positions *A* and –*A*. It can't go beyond these maximum positions because then you would have more spring potential energy than the total amount of energy that the system held initially, breaking the law of conservation of energy.

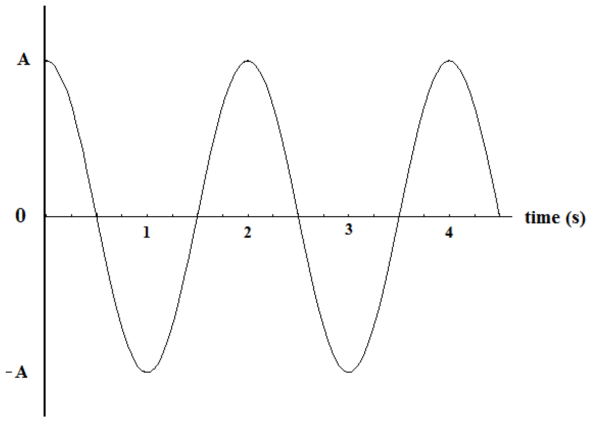
This maximum position is given the symbol *A* because it is the *amplitude* of oscillation.

The time it takes for one complete cycle of motion, from *A* to –*A* back to *A*, is the period of motion, *T*.

The number of oscillations per second is the frequency, *f*. So if it takes two seconds for every one single oscillation, then the block only moves one-half of an oscillation per second, making period and frequency reciprocals of each other.

$T = \frac{1}{f}$ and $f = \frac{1}{T}$ period begin measured in seconds, frequency in $\frac{oscillations}{second}$ or Hertz (Hz).

If we made a graph of positon versus time for the block, it would look like this:



Again, you can see the time it takes to go from *A* to –*A* back to *A* is 2s, making the frequency 0.5Hz.

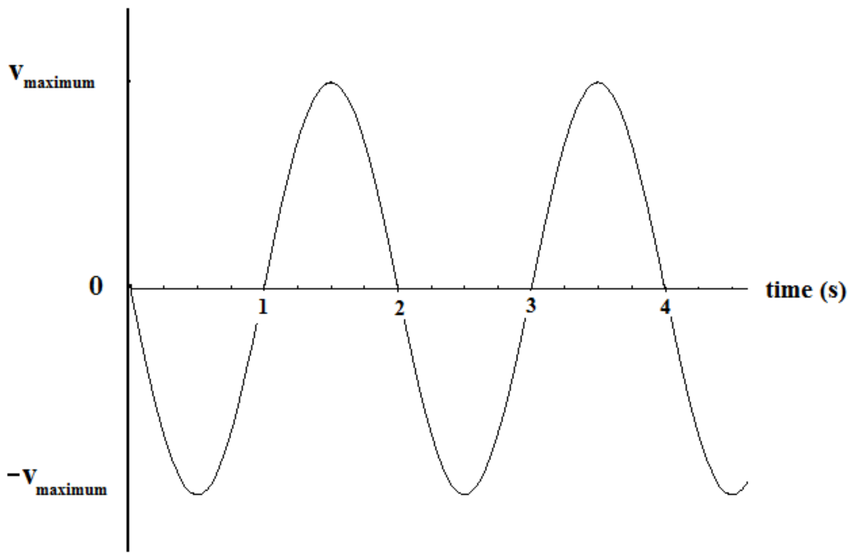
Answer Webassign Question 1

This type of oscillating motion that follows a sinusoidal curve is called *simple harmonic motion* and it will occur whenever the restoring force is proportional to the negative of the displacement from equilibrium. Here the spring force is the restoring force.

What is happening to the block in terms of velocity? Well, we know when it is released, it begins at rest. It is then pulled left by the stretched spring until it reaches the 0 position. During this time, it will pick-up leftward velocity. Once it moves left of the 0 position, the spring will be compressed, pushing it right and slowing it down. Once it reaches position –*A*, it will pause for an instant.

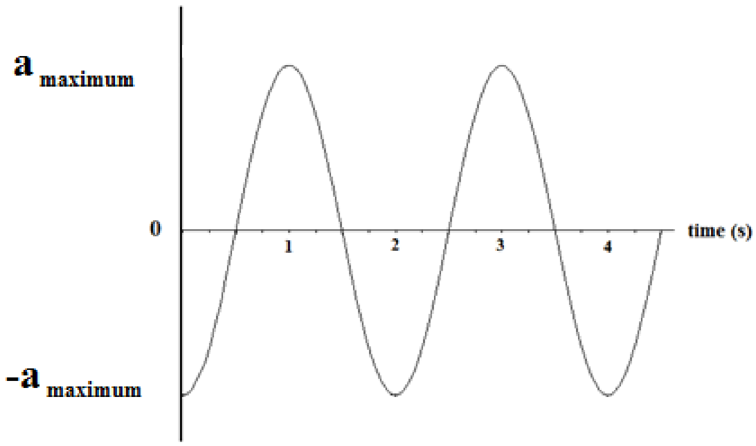
The block will then be pushed right, increasing in positive velocity until it reaches the 0 position. Once it moves right of that position, it will be pulled leftwards and slowed down. Upon reaching position *A*, it will pause for an instant and then the cycle will repeat itself.

All of this makes the following graph:



And for acceleration, we can just use the equations: $F = m\cdot a$ and $F_{spring} = -k\cdot x$

By the first equation, the acceleration will be whatever sign the force is; by the second equation, the spring force will have the opposite sign of the position. So the acceleration graph is just the position graph made negative, or turned upside-down:



An easy way of checking these is to use the old idea that velocity is the slope of a position-time graph and acceleration is the slope of a velocity-time graph.

If you look at the very start of the graphs, at time zero, the position graph has a slope of zero and the velocity graph has a value of zero. The velocity graph has a negative slope and the acceleration graph has a negative value.

At a time of 1.5s, the position slope is positive and the velocity value is positive. The velocity slope is zero and the acceleration value is zero.

This will work for any time you choose. It's also true that the area under the acceleration graph will match a change in velocity and the area under the velocity graph will match a change in position.

What are these values for maximum velocity and maximum acceleration?

We can find maximum velocity from conservation of energy. We begin with only spring potential energy and when the block reaches position zero, we have no spring potential left (if $U_{spring} = \frac{1}{2}kx^2$ and $x = 0$, $U_{spring} = 0$). Therefore, at positon zero, all of the spring energy has become kinetic energy.

$U_i + K_i = U_f + K_f$

$\frac{1}{2}kA^2 + 0 = 0 + \frac{1}{2}mv_{maximum}^2$

$v_{maximum} = A\cdot\sqrt{\frac{k}{m}}$

Answer Webassign Question 2

And maximum acceleration comes from, again, $F = m \cdot a$ and $F_{spring} = -k \cdot x$

$m \cdot a_{maximum} = k \cdot A$

$a_{maximum} = A \cdot \frac{k}{m}$

Answer Webassign Question 3

If you've studied trigonometry, you know that the position graph is a cosine curve, the velocity graph is a negative sine curve, and the acceleration graph is a negative cosine curve.

More specifically, if we define angular frequency as $\omega = \sqrt{\frac{k}{m}}$ we find that

For position: $x = A \cdot \cos(\omega \cdot t)$

For velocity: $v = - A \cdot \sqrt{\frac{k}{m}} \cdot \sin(\omega \cdot t)$

For acceleration: $a = - A \cdot \frac{k}{m} \cdot \cos(\omega \cdot t)$

Where these equations come from requires calculus, so I will leave all of that for an appendix. Without calculus, however, we know that $\omega = \frac{2\pi}{T}$. This simply comes the fact $\omega = \frac{\Delta\theta}{\Delta t}$ and 2π being the (numerator) angular displacement during the (denominator) time, *T*.

So $\sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ therefore $T = 2\pi\sqrt{\frac{m}{k}}$

Answer Webassign Question 4

And if $f = \frac{1}{T}$ then $f = \frac{1}{2\pi} \cdot \sqrt{\frac{k}{m}}$

Answer Webassign Question 5

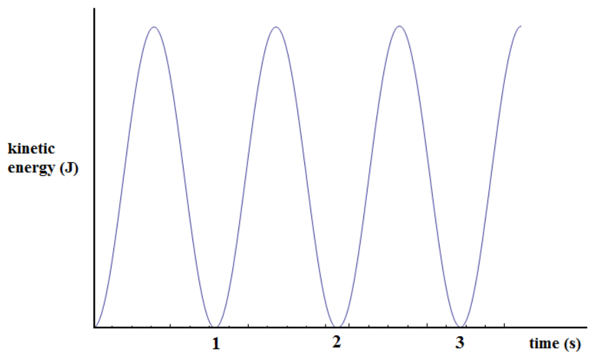
Using the equations $K = \frac{1}{2}mv^2$ and $U_{spring} = \frac{1}{2}kx^2$

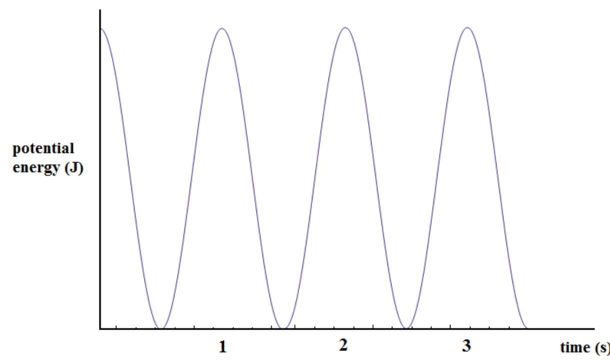
We can insert the equations for v and x into these two and get:

$K = \frac{1}{2}m[A\cdot\sqrt{\frac{k}{m}}\cdot\sin(\omega\cdot t)]^2$ or $K = \frac{1}{2}k\cdot A^2\cdot\sin^2(\omega\cdot t)$

$U_{spring} = \frac{1}{2}k[A\cdot\cos(\omega\cdot t)]^2$ of $U_{spring} = \frac{1}{2}k\cdot A^2\cdot\cos^2(\omega\cdot t)$

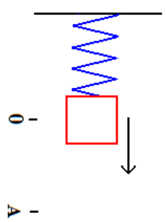
So the kinetic energy graph is a sine-squared graph and the spring potential energy graph is a cosine-squared graph. Those look like this:





Because of the v^2 and the x^2 values in the equations, these energies are always positive.

Suppose that instead of a block sliding left and right, you attached a block to a vertical spring and let it settle-down to rest. You then pull it down an additional distance *A* and release it.



The block will oscillate up and down around the zero position, much like the horizontal oscillator. And even though there is a force of gravity throughout, you can actually derive the same equations as the ones for a horizontal oscillator. This proof I will put at the end as another optional appendix.

To review:

The period of time is the time of one complete oscillation. For a spring mass oscillator, this is:

$T = 2\pi\sqrt{\frac{m}{k}}$

Frequency is the number of oscillations per second, measured in Hertz (Hz). This is:

$f = \frac{1}{2\pi}\cdot\sqrt{\frac{k}{m}}$

Period and frequency are reciprocals of each other.

The graphs of position, velocity, and acceleration will all be sinusoidal. Whether they are sine, cosine, negative sine, or negative cosine depends upon the initial conditions of the system.

The acceleration of the oscillator can be found using F = ma and $F_{spring} = -k\cdot x$

The velocity of the oscillator can be found using conservation of energy, knowing that:

$U_{spring} = \frac{1}{2}kx^2$ and $K = \frac{1}{2}mv^2$

Appendix 1 – derivation of oscillation functions

If F = ma and $F_{spring} = -k\cdot x$, then $ma = -k\cdot x$ where $a = \frac{d^2x}{dt^2}$

So $\frac{d^2x}{dt^2} = -\frac{k}{m}\cdot x$

What function of x solves this equation? Well, we know the actual motion looks like a cosine curve, so let's try that. Suppose $x = A\cdot\cos(\omega t)$.

If we differentiate, $v = -\omega A\cdot\sin(\omega t)$

If we differentiate again, $a = -\omega^2 A\cdot\cos(\omega t)$

Sitting inside this acceleration equation is the initial position equation, $x = A\cdot\cos(\omega t)$.

So we can write $a = -\omega^2\cdot x$ or $\frac{d^2x}{dt^2} = -\omega^2\cdot x$

Dynamically, we know $\frac{d^2x}{dt^2} = -\frac{k}{m}\cdot x$

So $\omega^2 = \frac{k}{m}$ and $\omega = \sqrt{\frac{k}{m}}$

Inserting this value back into the equations for x, v, and a yields the equations needing proof.

Appendix 2 – derivations for a vertical oscillator

Suppose the block has a mass *m* and the spring has a spring constant *k*.

When the block is attached and settles down, the net force will be zero.

$F_{gravity} + F_{spring} = 0$

$-mg + -ky = 0$

$y = \frac{-mg}{k}$

We will define this equilibrium position as the zero position and then pull the block down an additional displacement *A*. As it oscillates, when it has a position of y_2 relative to equilibrium, the net force of the block will be:

$\Sigma F = F_{spring} + F_{gravity} = -k(y + y_2) + (-mg) = -k(\frac{-mg}{k} + y_2) + (-mg) = -k \cdot y_2$

So $m \cdot a\ -k \cdot y_2$ where y_2 is the position from equilibrium, satisfying the differential equation which yields all of the oscillation functions.

The spring potential energy graph will be more complex, however, because $U_{spring} = \frac{1}{2} \cdot k \cdot (y + y_2)^2$

There will be a gravitational potential energy graph, $U_G = mg \cdot y_2 = mg \cdot A \cdot \cos(\omega t)$.

The kinetic energy graph will still be: $K = \frac{1}{2}\, k \cdot A^2 \cdot \sin^2(\omega \cdot t)$

If you work out the math, you will find a total energy, $E = \frac{k \cdot A^2}{2} + \frac{m^2 \cdot g^2}{2k}$ which is independent of time.