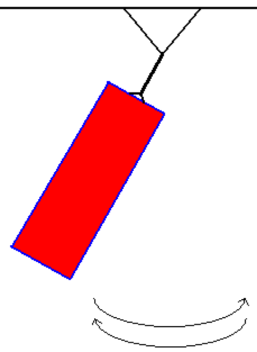
Pendulums



At a gym, you push a punching bag to the side and let it go. You find that is swings back and forth with an oscillatory motion.

This is called a physical pendulum and the time it takes for one back-and-forth motion is the period of oscillation:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

I is the rotational inertia of the punching bag around its point of rotation
m is the mass of the punching bag
g is the acceleration due to gravity
d is the distance between the point of rotation and the center of mass of the punching bag

Answer Webassign Question 1

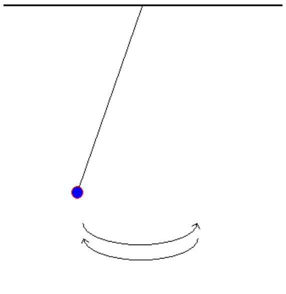
The derivation of this equation requires calculus, so I will leave it to an appendix.

As mentioned before, frequency is the reciprocal of the period, so

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{mgd}{I}}$$

Answer Webassign Question 2

If, instead, we simply have a very small ball bearing tied to a long string, this is called a simple pendulum.



Suppose the ball bearing has a mass *m* and the distance between the center of the ball bearing and the point of rotation is *L*.

Then $I = mL^2$ because the ball bearing is very nearly a point mass

And d = L

Putting these into the equation for a simple pendulum yields:

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

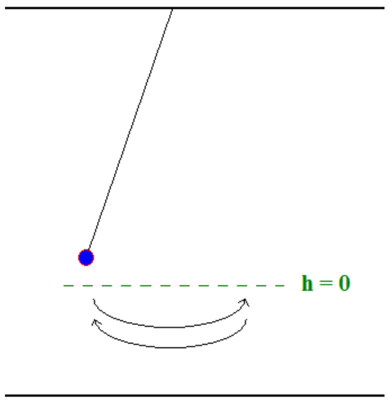
So this is the equation for the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}$$ where L is considered the length of the pendulum

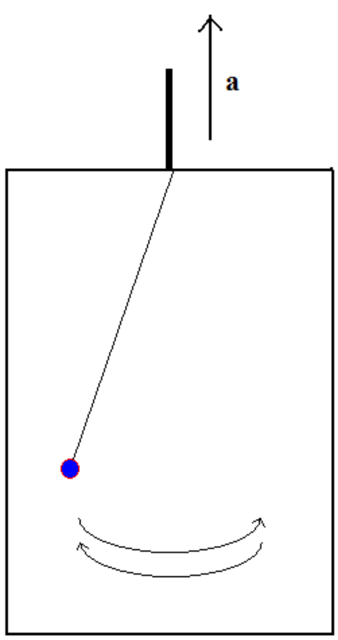
Answer Webassign Question 3

Therefore, $$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Answer Webassign Question 4



If we take the lowest position of the pendulum as it swings to be a height of zero, when it is at this position, it has zero gravitational potential energy. When it is at its maximum angular position from vertical, either on the left side or right, the pendulum pauses for a moment with maximum gravitational potential energy. Because energy is conserved, it has maximum kinetic at the base of the swing and zero kinetic at the two peaks of the swing.



If you put the pendulum in an elevator and the elevator accelerates, you can use the equations for a simple pendulum as long as you use the *effective* gravitational acceleration for g.

$$g_{effective} = g_{planet} + a$$

For instance, if the elevator had an acceleration of 2m/s^2 on the Earth, $g_{effective}$ = 11.8m/s^2.

Answer Webassign Question 5

Appendix – derivation of the physical pendulum period

This is going to be very similar mathematically to the derivation for a mass and spring oscillator. The main difference is that we use rotational concepts rather than linear concepts.

So, we begin with $\boldsymbol{\tau} = I \cdot \boldsymbol{\alpha}$ for the punching bag

By definition, $\alpha = \frac{d^2\theta}{dt^2}$

Because torque and center of mass are both linear equations, we can treat the force of gravity as if it acts on the center of mass of the punching bag.

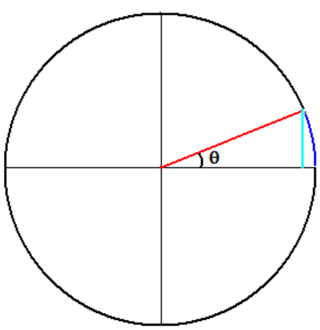
$\mathbf{R}_{cm} = \frac{m_1 \cdot R_1 + m_2 \cdot R_2 + m_3 \cdot R_3}{M}$ so $\mathbf{R}_{cm} \cdot \mathbf{g} = \frac{m_1 \cdot R_1 \cdot g + m_2 \cdot R_2 \cdot g + m_3 \cdot R_3 \cdot g}{M} = \frac{\Sigma\tau}{M}$ or $\Sigma\boldsymbol{\tau} = \mathbf{R}_{cm} \cdot m\mathbf{g} = \mathbf{R}_{cm} \cdot \mathbf{F}_G$

We'll call the distance to the center of mass simple *d*.

$|d| \cdot |mg| \cdot \sin\theta = I \cdot \frac{d^2\theta}{dt^2}$

To solve this differential equation in a relatively straightforward way, we need to use the small angle approximation which states that $\sin\theta \sim \theta$.

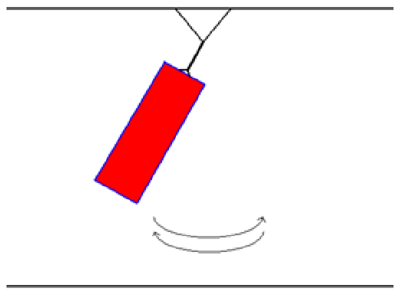
You can see this if you just look at a unit circle:

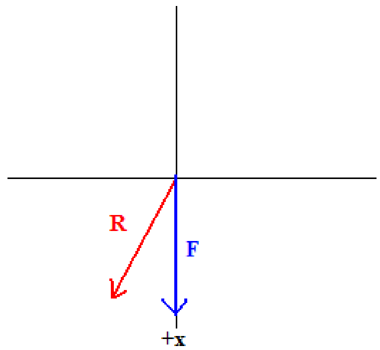


Because the radius is one, the length of the light blue line is $1 \cdot \sin\theta = \sin\theta$. Because the angle is measured in radians, the length of the dark blue line is $1 \cdot \theta = \theta$. (Arc length equals radius times the angle in radians). You can see, as the angle gets small, the light and dark blue lines become very close to the same length.

So now the equation becomes: $mgd \cdot \theta = I \cdot \frac{d^2\theta}{dt^2}$

However, if we call downward the positive x-axis, we see that a negative position angle will generate a positive torque and a positive angular acceleration:





So we need to introduce a negative sign: $-mgd\cdot\theta = I\cdot\frac{d^2\theta}{dt^2}$

Now, we predict that the angle of the pendulum will oscillate according to $\theta = -\theta_{max}\cdot\cos(\omega_a t)$

Where we use ω_a to indicate angular frequency, as opposed to the actual physical angular velocity of the swinging punching bag.

Differentiating once yields: $\omega = \omega_a\cdot\theta_{max}\cdot\sin(\omega_a t)$

Differentiating again yields: $\alpha = \omega_a^2\cdot\theta_{max}\cdot\cos(\omega_a t)$

Which has inside of it the angular position equation: $\theta = -\theta_{max}\cdot\cos(\omega_a t)$

So $\alpha = -\omega_a^2\cdot\theta$ or $\frac{d^2\theta}{dt^2} = -\omega_a^2\cdot\theta$

We know physically from before $-mgd\cdot\theta = I\cdot\frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\cdot\theta$

So $\omega_a^2 = \frac{mgd}{I}$

and the angular frequency, $\omega = \sqrt{\frac{mgd}{I}}$

As before, $\omega = \frac{2\pi}{T}$ so $\frac{2\pi}{T} = \sqrt{\frac{mgd}{I}}$ so $T = 2\pi\sqrt{\frac{I}{mgd}}$