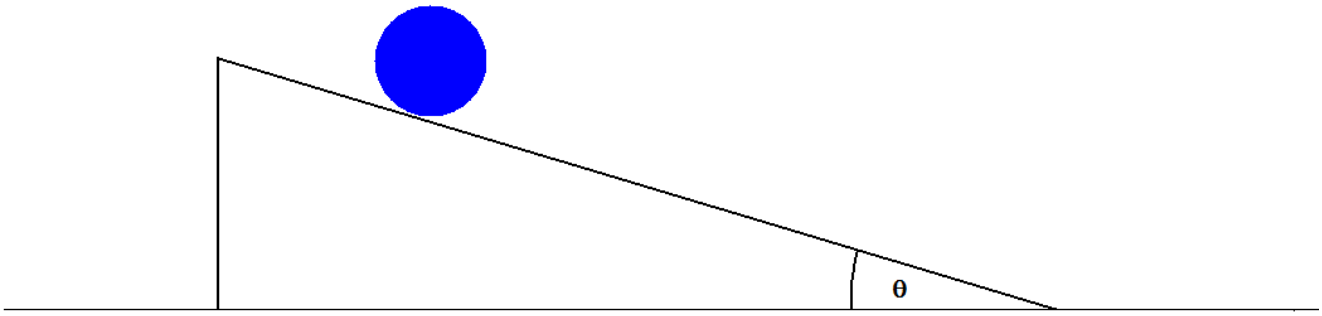
Rolling without slipping

A solid disk with mass *m* and radius *R* rolls down an incline sloped at an angle θ. What is the linear acceleration of the disk down the ramp?



Let's use conservation of energy and say the disk begins at rest with a height *h*. Taking the disk, ramp, and Earth as the system, there are no external forces, so no work is input.

By conservation of energy, the initial gravitational potential energy is becoming the kinetic energy of the disk. However, it is not just $mgh = \frac{1}{2}mv^2$.

At the bottom of the incline, the disk will have both translational kinetic energy ($\frac{1}{2}\cdot m\cdot v^2$) because it is moving left to right and rotational kinetic energy ($\frac{1}{2}\cdot I\cdot \omega^2$) because it is spinning around its center.

Therefore, $mgh = \frac{1}{2}\cdot m\cdot v^2 + \frac{1}{2}\cdot I\cdot \omega^2$

We can substitute $I = \frac{1}{2}\cdot m\cdot R^2$ for a disk and $\omega = \frac{v}{R}$ because we have rolling without slipping.

$\frac{1}{2}\cdot m\cdot v^2 + \frac{1}{2}\cdot(\frac{1}{2}\cdot m\cdot R^2)(\frac{v}{R})^2 = m\cdot g\cdot h$ and solving for v^2 produces

$v^2 = \frac{2}{1+\frac{1}{2}}\cdot g\cdot h$

From trigonometry, $\sin\theta = \frac{h}{\Delta s}$ where Δs is the displacement along the incline

so we can replace h with $\Delta s\cdot \sin\theta$ so that $v^2 = \frac{2}{1+\frac{1}{2}}\cdot g\cdot \Delta s\cdot \sin\theta$

The forces are not velocity-dependent, so the acceleration is constant and we can use $v_f^2 = v_i^2 + 2(a)(\Delta s)$

where $v_f^2 = \frac{2}{1+\frac{1}{2}}\cdot g\cdot \Delta s\cdot \sin\theta$ and $v_i^2 = 0$

$v_f^2 = v_i^2 + 2(a)(\Delta s)$ then becomes $\frac{2}{1+\frac{1}{2}}\cdot g\cdot \Delta s\cdot \sin\theta = 2(a)(\Delta s)$ and rearranging for acceleration

$$a = \frac{g \cdot sin\theta}{1 + \frac{1}{2}}$$

Answer Webassign Question 1

The $\frac{1}{2}$ in the denominator comes from the fact the object rolling was a disk with a rotational inertial of $\frac{1}{2}\cdot m\cdot R^2$. If the object was a hollow sphere, then $I = \frac{2}{3}\cdot m\cdot R^2$ and the acceleration would likewise change

$$a = \frac{g \cdot sin\theta}{1 + \frac{2}{3}}$$

Answer Webassign Question 2

Answer Webassign Question 3

Let's go back to the rolling disk and use

$$a = \frac{g \cdot sin\theta}{1 + \frac{1}{2}} = \frac{2}{3}\, g\cdot \sin\theta$$

If the disk was not rolling down the incline, but rather sliding down the incline, the acceleration would be simply $g\cdot\sin\theta$. Because it is instead a lesser $\frac{2}{3}g\cdot\sin\theta$, there must be some force pushing the rolling disk up the incline. This force is the same force causing the torque which gives the disk its clockwise angular acceleration as it rolls down the slope, namely static friction.

Using F = ma and taking down-the-slope as positive, we have:

$F_{\text{static friction}} + F_{\text{gravity, parallel to the incline}} = ma$

$F_{\text{static friction}} + mg\cdot\sin\theta = m\cdot(\frac{2}{3}g\cdot\sin\theta)$

Therefore, $F_{\text{static friction}} = -\frac{1}{3}(mg\cdot\sin\theta)$

Answer Webassign Question 4

Answer Webassign Question 5