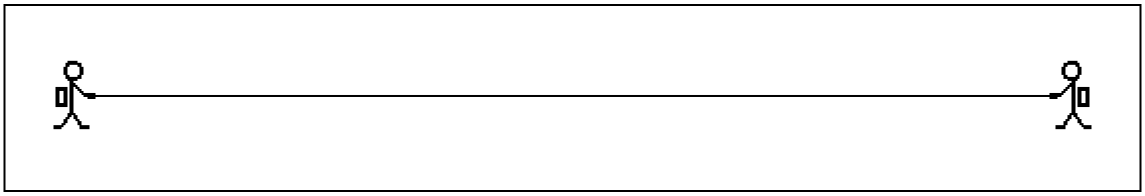
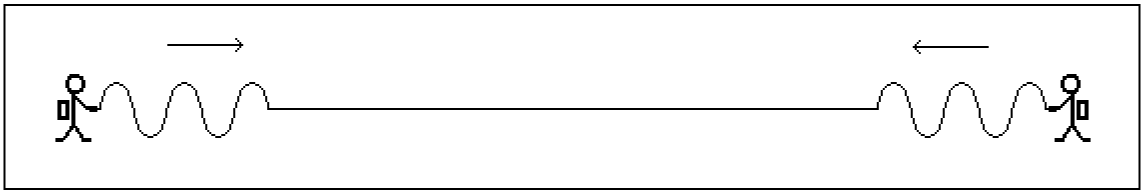
Standing Waves

Suppose two astronauts are floating out in free space and hold the ends of a rope:



Each astronaut oscillates their end of the rope up and down, creating the waves shown below:

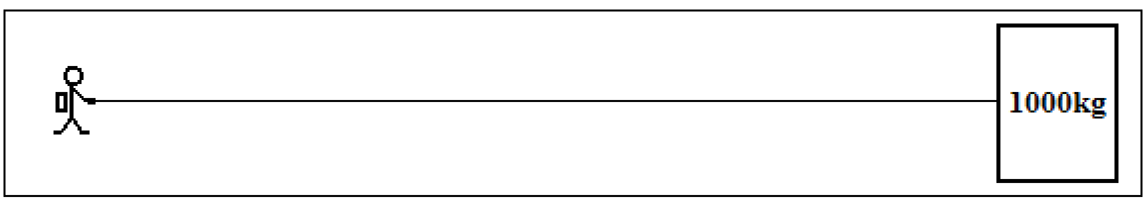


When these two waves reach each other and interact, they will follow the principle of superposition and create a standing wave as shown in the animation:

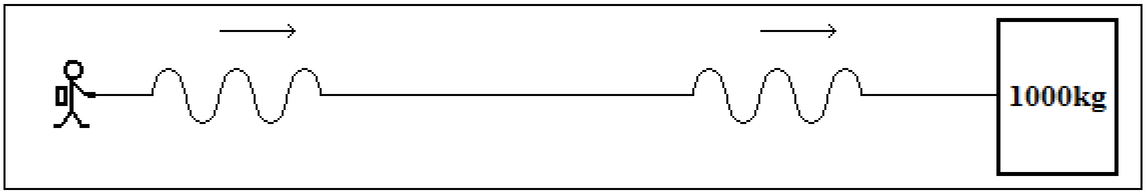
http://ophysics.com/w3.html

The red and blue animations represent the two waves sent by the two astronauts. The pink animation "standing wave" is the superposition sum of the red and blue waves. This can be shown mathematically, but the math is not particularly instructive, so I will leave it for an appendix.

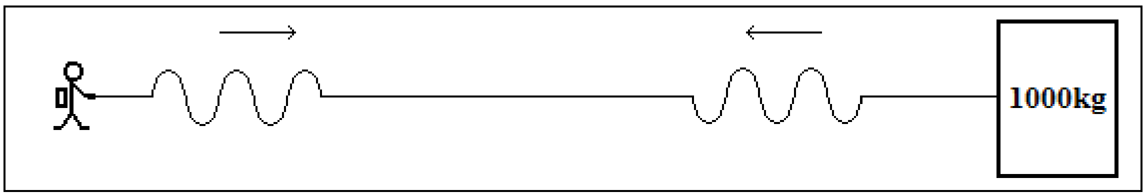
You can also achieve the same effect with only one astronaut if you replace the second astronaut with a very heavy wall:



The lone astronaut will send down two sets of waves.



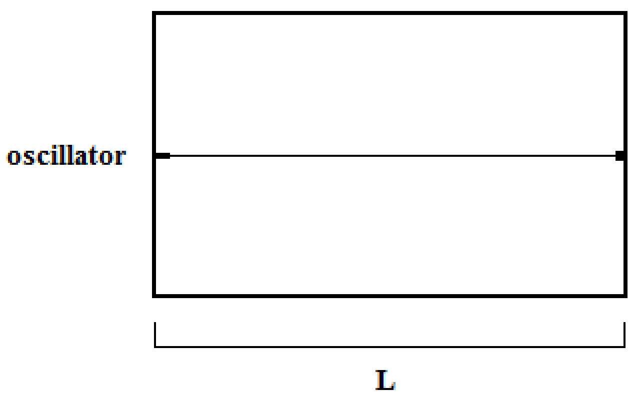
When the first set strikes the wall, it will be reflected and inverted, creating:



Again, when the two wave formations overlap, they will create the same "standing wave" that was created by the two astronauts.

And if the astronaut sends down a continuous stream of crests and troughs rather than just the segments shown above, the entire rope will eventually become one long standing wave.

Now let's do something similar, but a little more compact:



Above is a box with a width, *L*, with a rope that goes across the box. The right side is knotted to a fixed point and the left side is knotted to an oscillator that can rotate quickly between 90º and 270º, creating wave pulses in the rope.

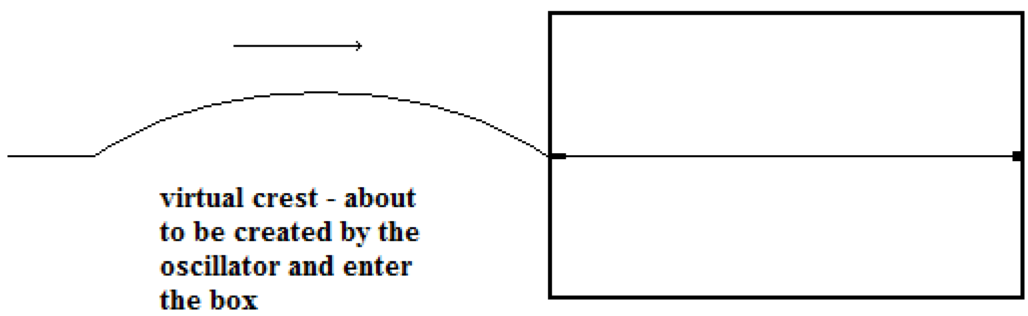
We can make standing waves inside the box, but we are not unlimited in the ones we can make because we have "boundary conditions". Those conditions are:

- whatever standing wave formation we make, it has a full length, *L*

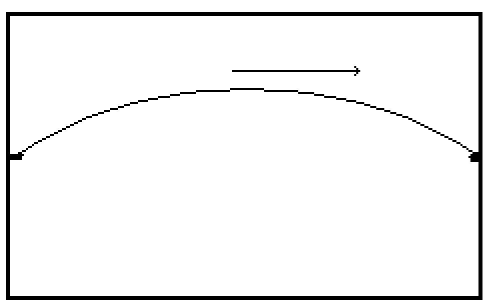
- the right side of the wave formation must remain stationary because the right end of our rope is knotted at a fixed position

- the left side of the wave formation must remain stationary because the left end of the rope is knotted to the oscillator which is not moving vertically, but rather just flicking the rope up and down

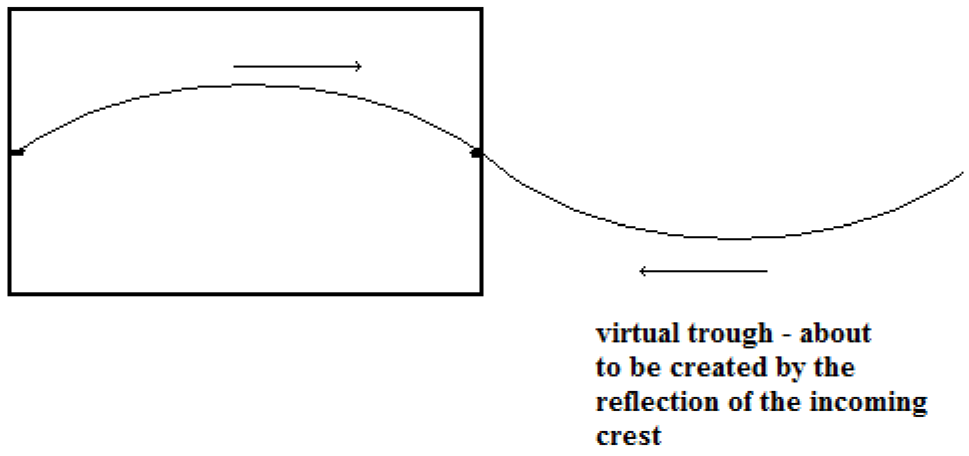
Here, the idea of a "virtual wave" become useful in visualizing what is happening. If the oscillator is going to create a single crest which travels from left to right, we can draw that "future crest" to the left of the oscillator as something traveling right and about to happen:



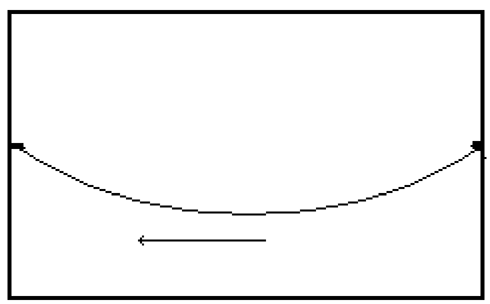
Once the virtual crest enters the box, it exists, having been created by the oscillator:



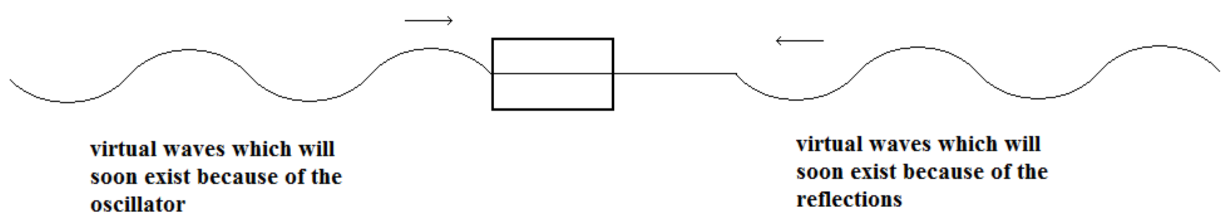
We know this crest is going to be reflected by the right side of the box and bounce left, inverted. We can diagram that too with a virtual wave:



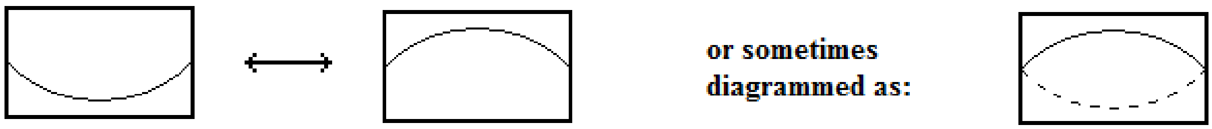
Then it enters the box and exists:



If we then switch the oscillator on and it runs continuously, there are indefinite streams of virtual waves coming in from left to right and virtual reflections coming in from right to left:



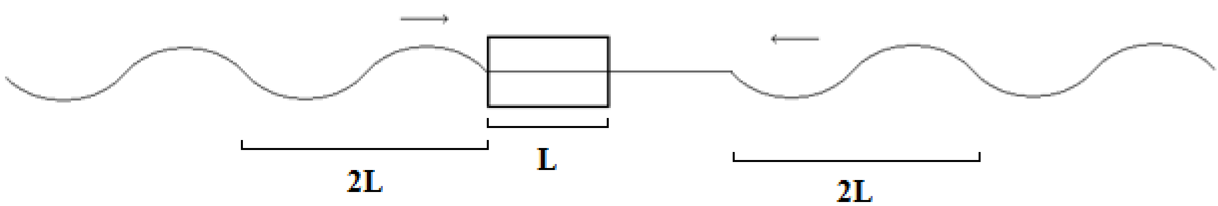
But this is just what we had with the two astronauts, so it is going to give us the same pink standing wave pattern inside the box as in the animation link. Inside the box, you would see the center of the rope bob up and down while the two ends remain fixed:



The middle of the rope oscillates between these two extremes

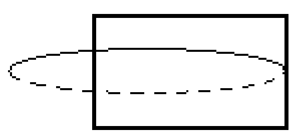
It's a little deceptive, because a rope bobbing up and down seems like a simple phenomenon when it's really a fairly complex summation of two interacting waves.

By the way, the standing wave above is said to have a wavelength of 2*L* because the incoming virtual waves that create it have a wavelength of 2*L*:



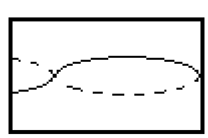
Here's an animation of this superposition of incoming waves and reflected waves:
http://www.walter-fendt.de/html5/phen/standingwavereflection_en.htm

However, we can't use the oscillator to create waves of any random wavelength. Remember, we are restricted by the boundary conditions. Suppose for example, we tried to make a standing wave with waves of wavelength 3*L*. That would look something like this:



which is no good, because it would require the oscillator on the left side of the box to slide up and down and that breaks one of the boundary conditions.

Same story if you tried to create a standing wave with waves of a wavelength 1.5*L*:



We can, however "fit inside the box" standing waves with wavelengths of:

$\lambda_1 = 2L$ $\lambda_2 = L$ $\lambda_3 = \frac{2}{3}L$ $\lambda_4 = \frac{1}{2}L$ $\lambda_5 = \frac{2}{5}L$ etc.

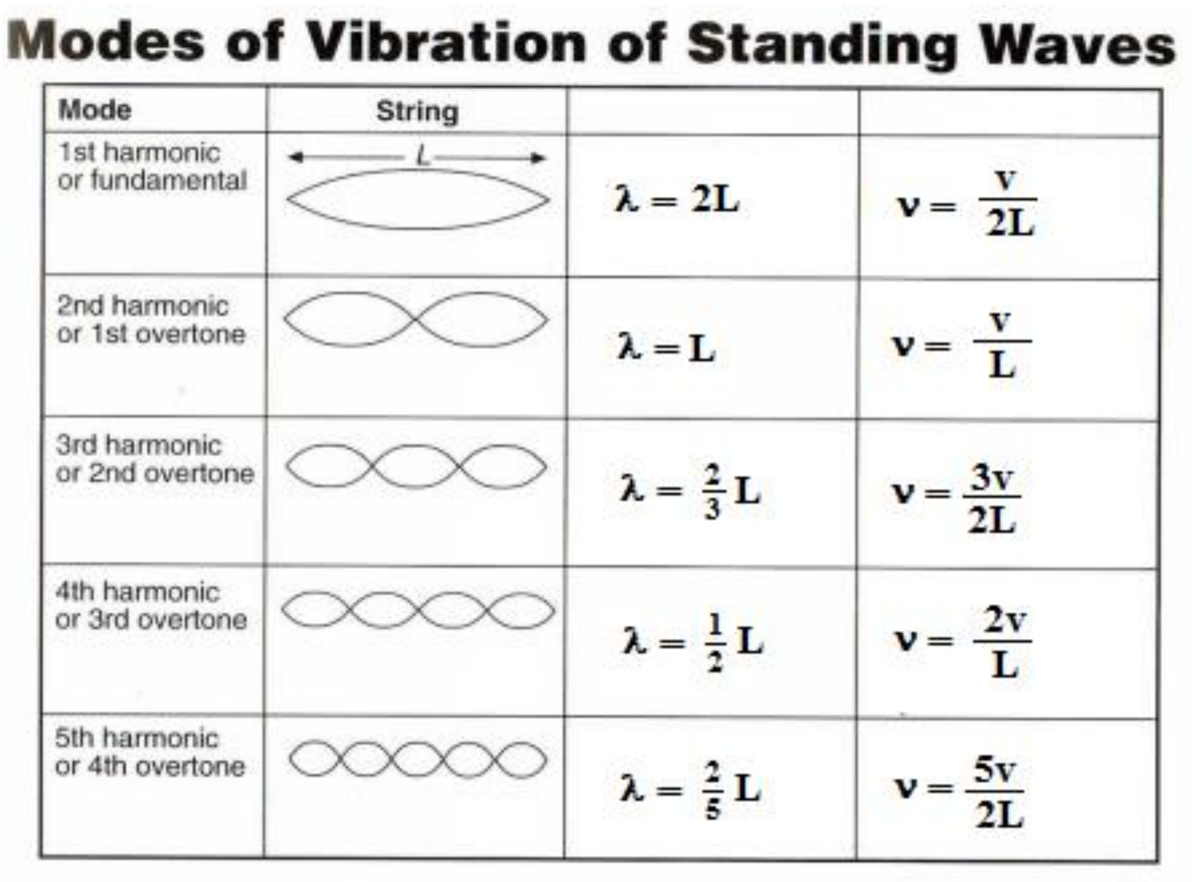
The pattern being:

$\lambda_n = \frac{2}{n}L$ for any n^{th} harmonic, where n = 1, 2, 3, etc.

If these standing waves are being created by virtual traveling waves which follow the equation $v = \lambda \cdot \upsilon$, then:

$v = \frac{2}{n}L \cdot \upsilon$ or $\upsilon_n = \frac{n \cdot v}{2L}$

Here's a standard textbook picture showing those:

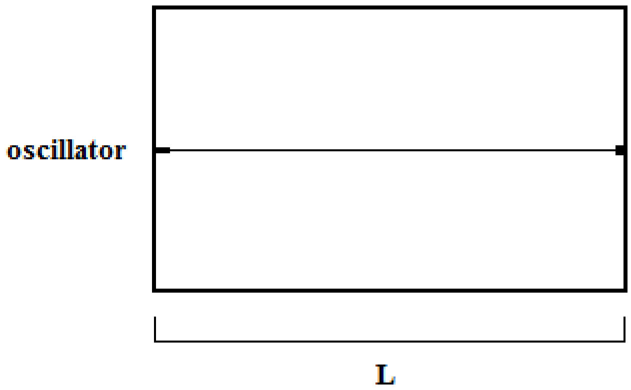


Modes of Vibration of Standing Waves

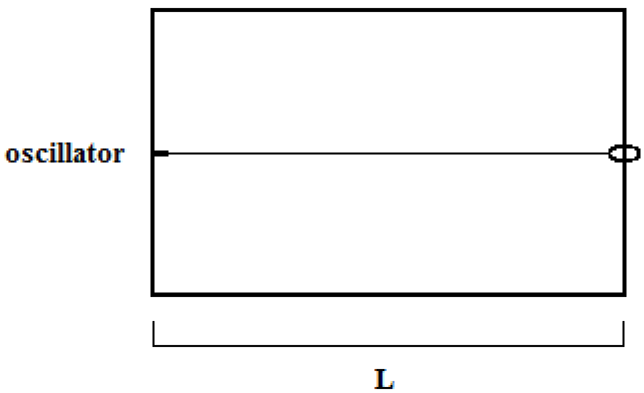
Mode	String		
1st harmonic or fundamental	L	$\lambda = 2L$	$\nu = \frac{v}{2L}$
2nd harmonic or 1st overtone		$\lambda = L$	$\nu = \frac{v}{L}$
3rd harmonic or 2nd overtone		$\lambda = \frac{2}{3}L$	$\nu = \frac{3v}{2L}$
4th harmonic or 3rd overtone		$\lambda = \frac{1}{2}L$	$\nu = \frac{2v}{L}$
5th harmonic or 4th overtone		$\lambda = \frac{2}{5}L$	$\nu = \frac{5v}{2L}$

Use this chart to answer Webassign questions 1 - 3

Now suppose we take our box with the rope going across the middle and the oscillator on the left side and make one modification:



Instead of the right side of the rope being knotted to a fixed point, we are going to tie it to a ring which can slide up and down a post without friction:



Our left side is still a fixed boundary, the rope knotted to the oscillator. But our right side is now a “free boundary”, free to slide vertically along the post to positions +A and –A, where A is the amplitude of the waves.

Now our boundary conditions are:

- whatever standing wave formation we make, it has a full length, *L*

- the right side of the wave formation must oscillate up and down between +A and –A, as would an oscillator attached to a spring. The little bit of rope on the right end is the oscillator’s mass and the vertical component of the rope’s tension provides the restoring force.

- the left side of the wave formation must remain stationary because the left end of the rope is knotted to the oscillator which is not moving vertically, but rather just flicking the rope up and down

The wave formations that now "fit inside the box" have wavelengths of:

$\lambda_1 = 4L$ $\qquad \lambda_3 = \frac{4}{3} L$ $\qquad \lambda_5 = \frac{4}{5} L$ $\qquad$ etc.

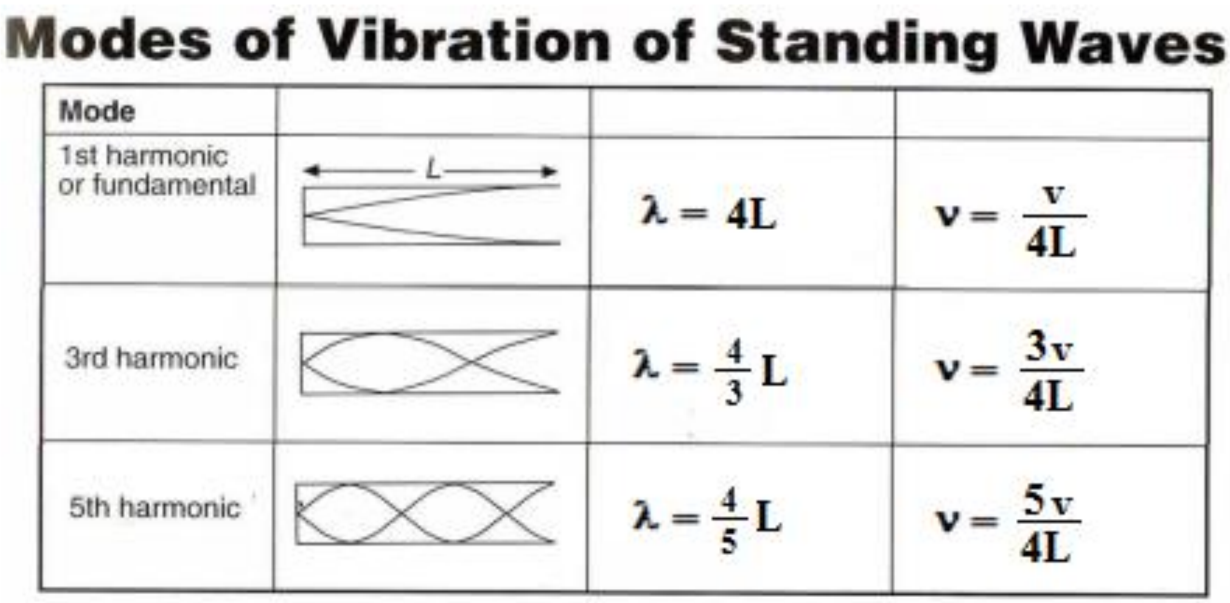
The pattern being:

$\lambda_n = \frac{4}{n} L$ $\qquad$ for any n^{th} harmonic, where n = 1, 3, 5, etc.

If these standing waves are being created by virtual traveling waves which follow the equation $v = \lambda \cdot \upsilon$, then:

$v = \frac{4}{n} L \cdot \upsilon$ $\qquad$ or $\qquad \upsilon_n = \frac{n \cdot v}{4L}$

Here's a standard textbook picture showing those:



Modes of Vibration of Standing Waves

Mode			
1st harmonic or fundamental		$\lambda = 4L$	$\nu = \frac{v}{4L}$
3rd harmonic		$\lambda = \frac{4}{3}L$	$\nu = \frac{3v}{4L}$
5th harmonic		$\lambda = \frac{4}{5}L$	$\nu = \frac{5v}{4L}$

Use this chart to answer Webassign questions 4 - 5

One question you may reasonably ask is: When you play a stringed instrument, how does the string "know" to vibrate with only the waves that satisfy the boundary conditions? The answer is that it doesn't. The string begins with a variety of wavelength phenomena, but only those wavelengths which satisfy the boundary conditions resonate with the body of the instrument and sustain their vibration without significant amplitude damping.

Appendix:

The general equation for a traveling wave is:

$y_1 = A\sin(kx - \omega t)$ for a wave traveling to the right

and

$y_2 = A\sin(kx + \omega t)$ for a wave traveling to the left

Here, we assume the frequency and wavelengths of the two incoming virtual waves are the same, where k is the wavenumber, $k = \frac{2\pi}{\lambda}$.

We can use the trigonometric identity:

$\sin\alpha + \sin\beta = 2\cos\frac{\alpha - \beta}{2} \cdot \sin\frac{\alpha+\beta}{2}$ to produce

$y_2 + y_1 = A\,[2\cos(\omega t)\cdot\sin(kx)]$ which is a standing wave oscillating with time where the frequency of oscillation is the same as the frequency of the component waves

To find the nodes of the standing wave, we have $y_1 + y_2 = 0$, so $\sin(kx) = 0$

$kx = \sin^{-1}(0) = n\cdot\pi$ where n = 0, 1, 2, 3, etc.

If $k = \frac{2\pi}{\lambda} = \frac{n\cdot\pi}{x}$ then $x = n\cdot\frac{\lambda}{2}$ which is to say, a node exists every half-wavelength