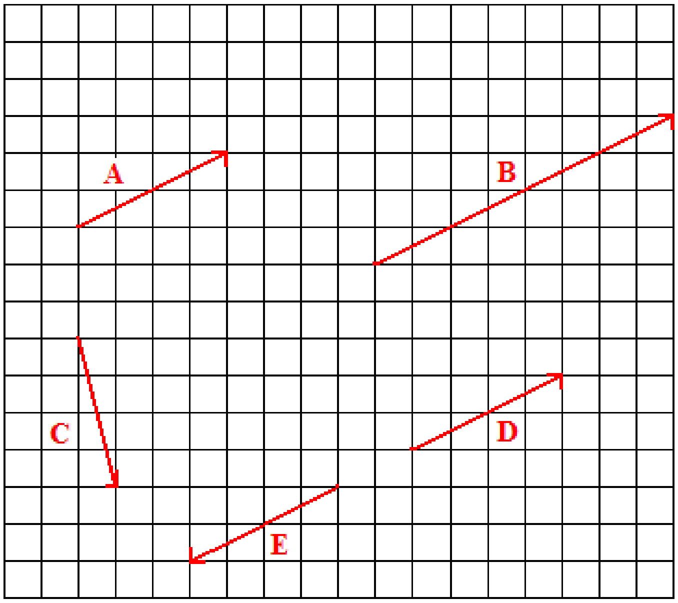
Torque



We've seen one way of multiplying vectors is the dot product or the scalar product. For example, **A▪C** = $A_xC_x + A_yC_y$ = (4)(1) + (2)(-4) = -4

It can also be calculated: **A▪C** = |A|·|C|·cosθ = (4.47)(4.12)(cos257.5º) = -4

Answer Webassign Question 1

Another way of multiplying vectors is called the vector cross product, or simply cross product.

One way of finding this is with the determinant of a 3x3 matrix, but I will leave that until the end as an optional appendix.

Instead, we'll use an equation similar to one for dot products. For example, the cross product of **A** and **C**, stated "A cross C" and written **A x C** equals |A|·|C|·sinθ. Again, |A| is the length of the **A** vector which can be found by the Pythagorean Theorem, |C| is the length of the **C** vector, and θ is the angle *from* the first vector *to* the second vector, swept counter-clockwise.

If you use inverse tangent, you will find **A** has an angle of 26.565º and **C** has an angle of 284º. So, to sweep from **A** to **C** counter-clockwise would be to sweep an angle of about 257.5º.

Therefore, **A x C =** |A|·|C|·sinθ = (4.47)(4.12)(sin 257.5º) = -18

The cross product has the highest magnitude when θ is either 90º or 270º, so, conceptually, the cross product is a measure of how perpendicular two vectors are. (Recall, the dot product was a measure of how parallel two vectors are.)

Answer Webassign Question 2

There is another difference between the two. While the result of the dot product is a scalar, the result of the cross product is a vector, meaning it has a direction. So what is the direction of our **A x C** result? To answer this, you must use the *right hand rule for cross products*. What you do is this:

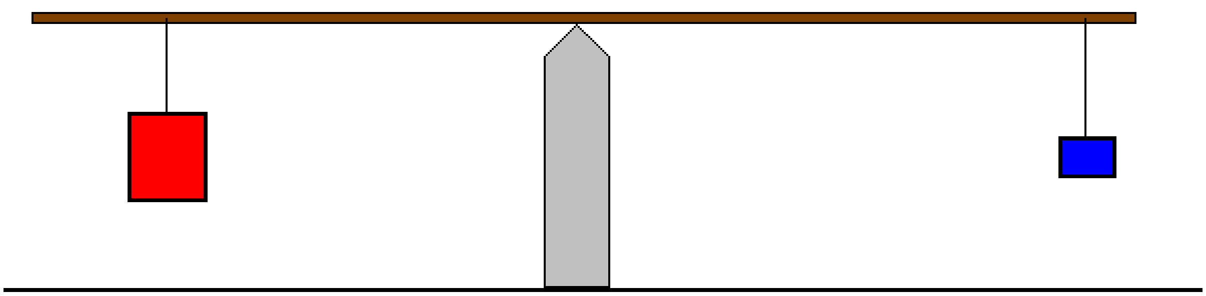
Physically take your right hand and, with palm flat, align the fingers with the first vector of the cross product. In our example, this is vector **A**, so you align your fingers with **A**, pointing them at about 26º. Then you sweep your fingers through the smaller path to hit the second vector. So you have a choice between sweeping your fingers clockwise or counter-clockwise to hit the **C** vector, and the shorter path is clockwise. When you make this sweep, you look at the direction of your thumb and this is the direction of the cross product. In this case, your thumb points into the screen, which is the negative z-axis. This agrees with the negative sign in our answer of -18.

Note that if you were finding **C x A**, you would sweep from **C** to **A** along the shortest path. This would be counter-clockwise, your thumb would point out of the screen along the positive z-axis, and the cross product would be positive.

Understanding the cross product is valuable, because the concept of torque is a cross product. By definition, torque is the cross product of radius with force and is symbolized with the Greek letter tau.

$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} = |R| \cdot |F| \cdot \sin\theta$$

Let’s use it in a few examples.

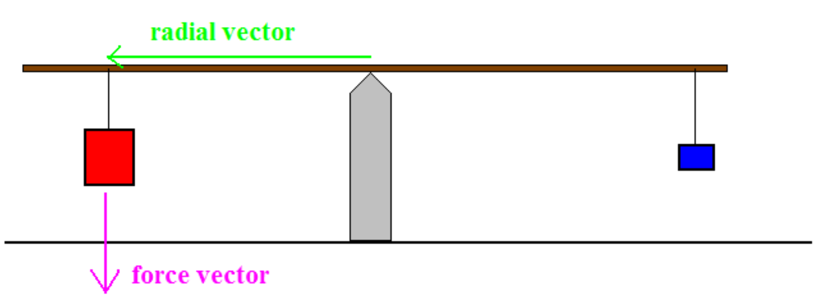


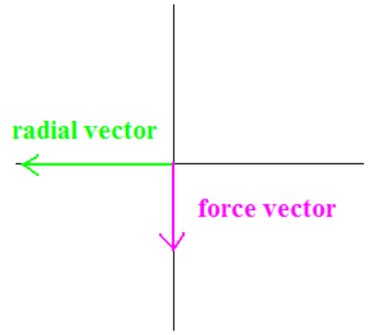
In the diagram above, a very light bar is balanced on a fulcrum. Let’s say the bar is 2m in length. A red mass of 3kg hangs 80cm to the left of the fulcrum and a 2kg blue mass hangs 95cm to the right of the fulcrum. What are the two torques applied?

Take the red mass first. The radius in the torque equation is the distance between the point of rotation and where the force is being applied. Well, the point of rotation is the fulcrum and the force is being applied to the bar where the cord is tied, 80cm from that fulcrum. So |R| = 0.8m.

The force at that point on the bar is going to come from the tension in the cord, which would be 29.4N, so |F| = 29.4N.

What is the angle to use in $\sin\theta$? Well, it can help to draw the vectors for radius and force.





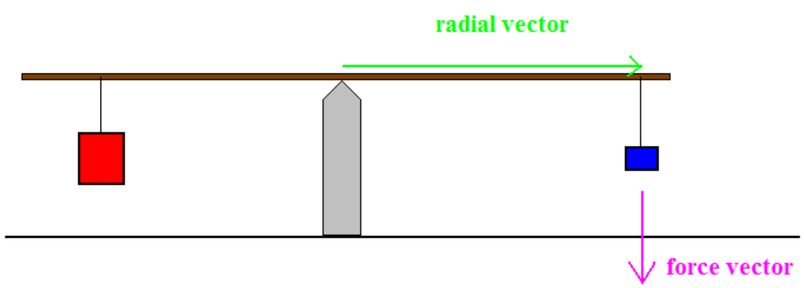
The radius, or radial vector, point *from* the point of rotation *to* where the force is applied. For the tension of the red block this is to the left, or 180º. The force vector is down, or 270º, because the cord pulls down on the thin beam.

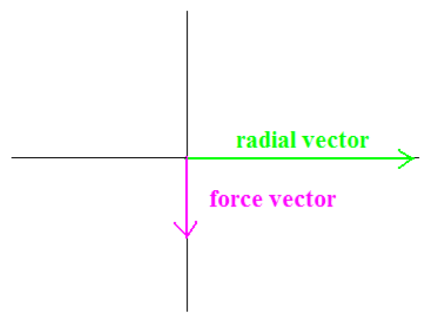
Again the θ of $\sin\theta$ is the angle *from* the first vector of the cross product (R) *to* the second vector (F), swept counter-clockwise. If you look at the vector diagram on the right, you can see this is 90º.

So, $\boldsymbol{\tau}_{\text{red block}} = (0.8\text{m})(29.4\text{N})(\sin 90^\circ) = 23.52\text{m}\cdot\text{N}$

In what direction is this torque? Point your right hand with the radial vector, sweep your fingers to the force vector along the shorter path, and see which way your thumb points. It should point out of the page, along the +z axis, matching the positive value we calculated from the equation.

For the torque applied by the blue mass,





$\boldsymbol{\tau}_{\text{blue block}} = (0.95\text{m})(19.6\text{N})(\sin 270^\circ) = -18.62\text{m}\cdot\text{N}$

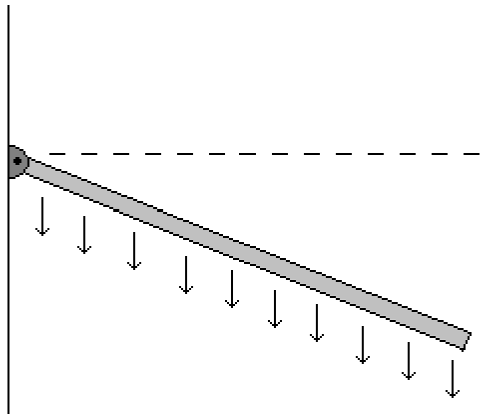
The net torque is simply the sum of the two torques, $\Sigma\boldsymbol{\tau} = 4.9\text{m}\cdot\text{N}$

Because the net torque is positive, it will cause a positive angular acceleration, which is a counter-clockwise angular acceleration.

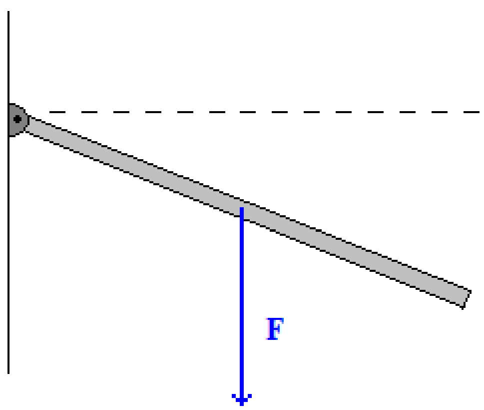
Answer Webassign Question 3

Answer Webassign Question 4

What if we have a force that is not localized at a particular radius? For example, what is the torque from the force of gravity on a beam 30º below the x-axis?



The Earth pulls down on every little piece of the beam with a force of *-mg*, but every little piece of the beam has a different radius from the hinge. Fortunately, torque is a linear equation, so we can just imagine that all of the mass is localized at the center of the beam and use that distance in the torque equation.



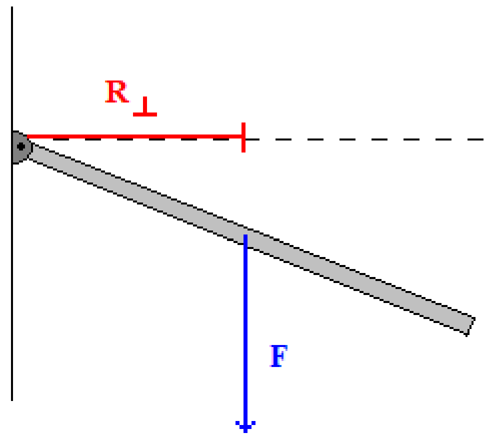
For example, if the beam was 5kg in mass, 2.6m in length, and 30° below the x-axis, the torque would be:

$\tau = (1.3\text{m})(49\text{N})(\sin 300^{\circ}) = -55.166\ \text{m}\cdot\text{N}$

Answer Webassign Question 5

As with work and the dot product, there is a less-technically-correct way of conceptualizing torque, which is to consider the force times the radius perpendicular to that force.

$\tau = R_{\perp} \cdot F$



In the same example, the effective radius is 1.3m, half the length of the beam. The perpendicular radius is then (1.3m)(cos 30º) = 1.126m. Multiply this by the force of gravity magnitude (49N) and the torque of 55.166 m·N is again found. When using this shortcut, you will often need to add the correct sign; here, that is negative because the torque is clockwise.

This perpendicular radius is often called the *lever arm* of the torque.

To review:

$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} = |R| \cdot |F| \cdot \sin\theta$$

R is the distance between the point around which the system rotates and the point at which the force is applied

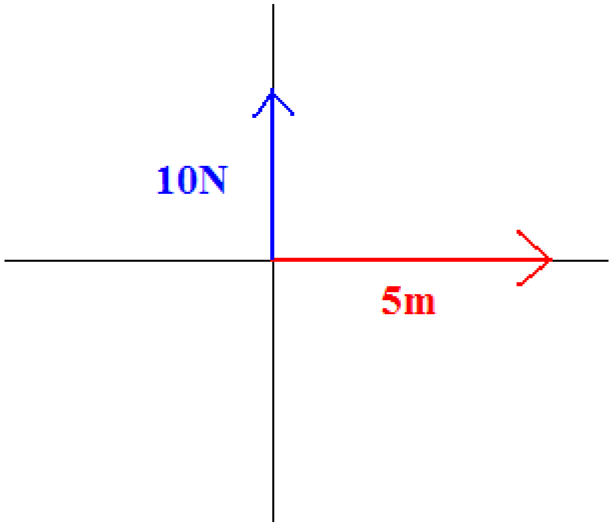
F is the force applied at that point

θ is the angle swept out from the **R** vector to the **F** vector, counter-clockwise. The **R** vector points from the point of rotation to where the force is applied. The **F** vector points in the direction in which the force is applied.

If the radii and forces are in the xy-plane, then the torques will be along the z-axis, either into (-) or out of (+) the screen. The right hand rule for cross products can confirm the correct sign.

Appendix – cross product as a matrix determinant

Let's take a simple example:



Here the force vector is 10N in the y-dimension and the radial vector is 5m in the x-dimension.

Written as a matrix, the cross product is:

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R_x & R_y & R_z \\ F_x & F_y & F_z \end{matrix}$$

Filled-in with the example above, it would become:

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5m & 0 & 0 \\ 0 & 10N & 0 \end{matrix}$$

The determinant is simply (50m·N)$\hat{k}$ and this is the torque.