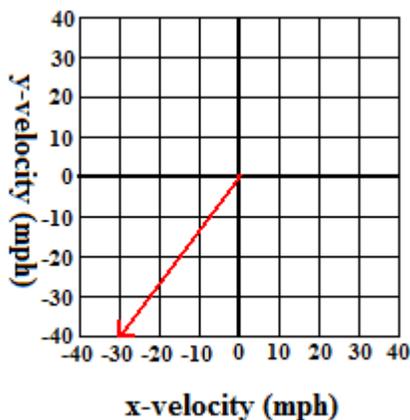# Vectors

Extending the concepts of kinematics into two and three dimensions, the idea of a vector becomes very useful. By definition, a vector is a quantity with both a magnitude and a spatial direction. This is opposed to a scalar quantity which has only magnitude. For example, 5mph north is a vector because it differs from 5mph south. But 98.6°F is a scalar because it cannot make reference to a direction in any reasonable way.

In kinematics, the simplest concept is position, so let's begin with a position vector shown below:
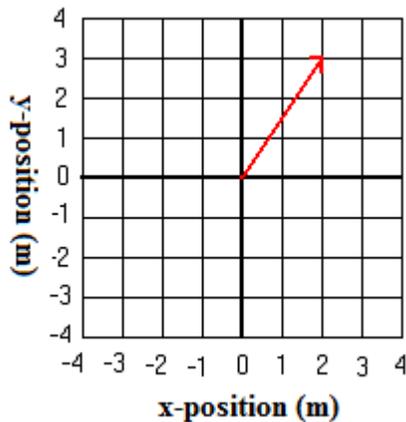
x-position (m)

If you were standing at the tip of the red arrow, relative to the origin, we could say you have a position of 2m in the x-dimension and 3m in the y-dimension. This is more succinctly written as $\mathbf{s} = < 2,3 >$m or $\vec{s} = < 2,3 >$m, where both the bold face and little arrow indicate the quantity is a vector. The red arrow in the diagram above is a visual representation of this vector.

The idea works just as well with velocity. Suppose you were driving simultaneously 30mph west and 40mph south. Your velocity vector would look like this:

x-velocity (mph)

and would be written as $\mathbf{v} = < -30, -40 >$mph or $\vec{v} = < -30, -40 >$mph. Writing the vector this way is writing it in what are known as Cartesian coordinates.

An alternate method is to write the vector in what are known as polar coordinates. This notation also has two numbers, the first being the magnitude of the vector and the second being its angle relative to the positive x-axis.

Take the velocity vector above. How long is that vector? Well, one can turn the vector into the hypotenuse of a right triangle and then use the Pythagorean Theorem.

x-velocity (mph)
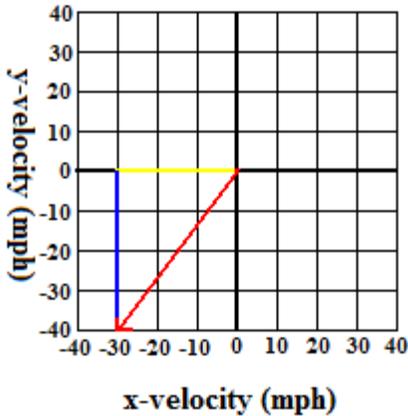
Here $a^2 + b^2 = c^2$ becomes $v_x^2 + v_y^2 = v^2$ where v is the length of the vector, the magnitude of the velocity, also known as the speed. Filling in the equation yields $(-30)^2 + (-40)^2 = v^2$, so v = 50mph.

But that single number of 50mph doesn't specify the direction in which it points relative to the origin. To specify that, we notate the angle the vector makes from the positive x-axis.

x-velocity (mph)
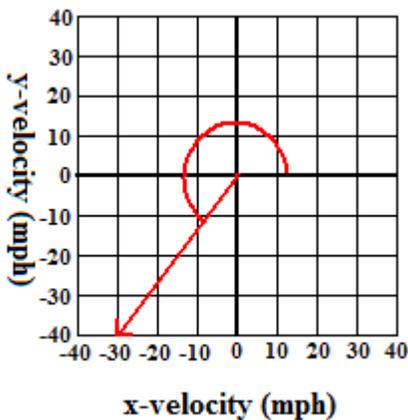
As seen above, it's how far you must sweep counter-clockwise from the positive x-axis until you hit the vector you are describing. Clearly, the angle is going to be something between 180° and 270°. To determine the specific value, you must use the tangent function. By definition, the tangent of the angle is equal to the y-component divided by the x-component. This is written as:

$$\tan \theta = \frac{v_y}{v_x}$$

To solve this, take the inverse tangent of both sides:

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{-40}{-30}\right) \qquad \text{which becomes } \theta = \tan^{-1}\left(\frac{-40}{-30}\right)$$

Type this into your calculator (in degree mode since you are working with degree units) and you will probably get $\theta = 53.13^\circ$. The problem is that the diagram clearly shows an angle between 180° and 270°. What's happening is that your calculator can't tell the difference between $\tan^{-1}\left(\frac{-40}{-30}\right)$ and $\tan^{-1}\left(\frac{40}{30}\right)$ and it's giving you the answer to the second by default. To correct this, you must add 180° to the calculator output, making the answer 233.13°.

Now the vector is completely specified in polar coordinates, $\mathbf{v} = (50\text{mph}, 233.13^\circ)$. Again, the first number is the magnitude or length of the vector, the second is the angle relative to the positive x-axis.

You will need to add the correction factor of $+180^\circ$ to your calculator output whenever the vector is in the second or third quadrants. If the vector is in the fourth quadrant, you calculator will give you an answer like -20°, which is the angle clockwise from the positive x-axis. But it's better to write the angle as counter-clockwise from the positive x-axis, so simply add 360° to make this $+340^\circ$.

Answer Webassign Question 1

Now suppose you had a vector with known polar coordinates, **a** = (12m/s², 120º) and you wanted to convert this to Cartesian coordinates.
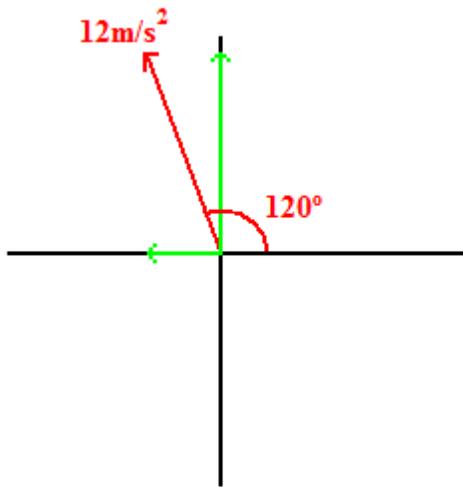

What you know is the length of the diagonal (12m/s²) and its angle relative to the positive x-axis. What you want to calculate are the x and y components of that vector, shown in the diagram as the green "shadows" projected onto the x and y axes.

To do this, use the sine and cosine functions. By definition, the x-component is equal to the vector magnitude times cosine of the vector angle. So a_x = (12m/s²)(cos 120º) = -6m/s²

And the y-component is equal to the vector magnitude times sine of the vector angle. So a_y = (12m/s²)(sin 120º) = 10.39m/s². You can see these two values roughly match the green shadow lengths.
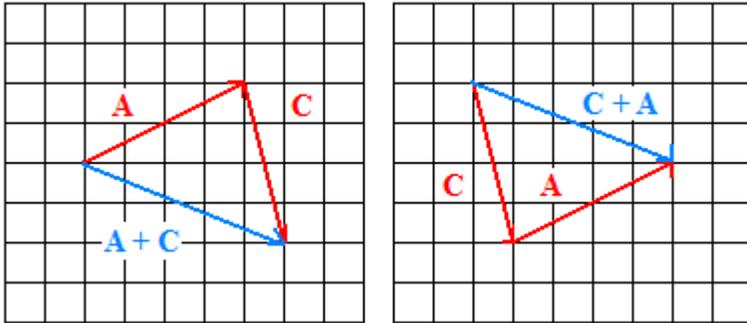
Answer Webassign Question 2
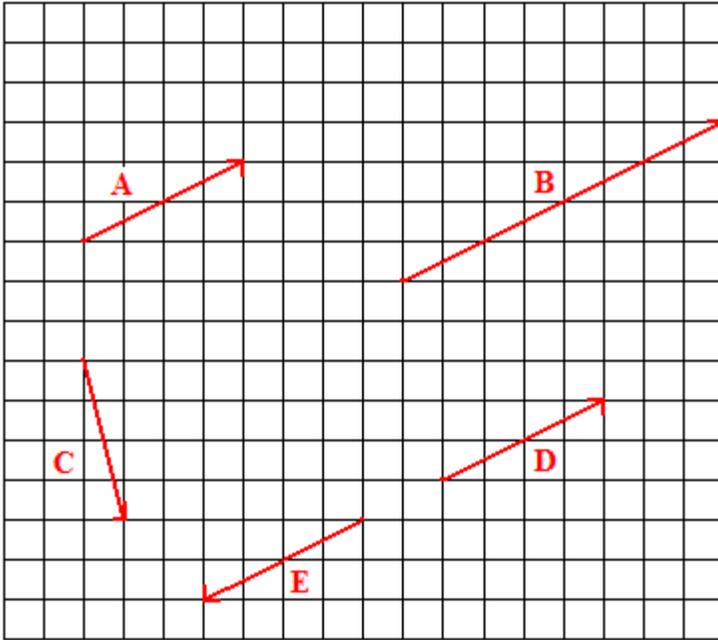
A few points about vectors that will use the diagram above:

1. Vector **A** can be written as $< 4,2 >$ and so can vector **D**. So even though they exist at different points in the diagram, they are entirely equivalent mathematically.

2. Multiplying a vector by a number is fairly easy. If you multiplied vector **A** by two, you would simply get $2 \cdot < 4, 2 > = < 8, 4 >$. This is the same as vector **B**, so you could write $2\mathbf{A} = \mathbf{B}$.

3. You can also multiply a vector by a negative number. For instance vector **A** times -1 would be written $-1 \cdot < 4, 2 > = < -4, -2 >$. This is the same as vector **E**, so you could write $-\mathbf{A} = \mathbf{E}$. Visually, the negative of a vector is simply the vector rotated 180º.

4. Suppose you wanted to add vectors **A** and **C**. Mathematically, this would simply be written $\mathbf{A} + \mathbf{C} = < 4,2 > + < 1, -4 > = < 5, -2 >$. You're simply adding the x-components and then the y-components. Visually, it is done by drawing the first vector and then drawing the tail of the second vector off the tip of the first.

Answer Webassign Question 3
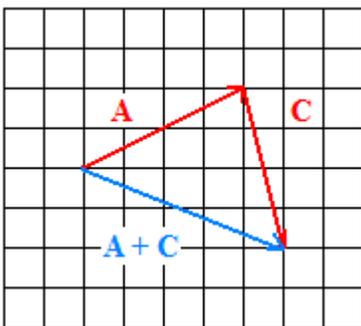
It's also fairly easy to see that **A** + **C** is the same as **C** + **A**.

To review, vectors can be written in Cartesian and polar coordinates.

To translate from Cartesian to polar velocity, use the Pythagorean theorem $v = \sqrt{v_x^2 + v_y^2}$ and $\tan^{-1}(\frac{v_y}{v_x})$. Remember to add 180° in the second and third quadrants, 360° in the fourth.

To translate from polar to Cartesian velocity, use $v_x = v \cdot \cos\theta$ and $v_y = v \cdot \sin\theta$.

For problems in two dimensions, the best idea is to convert everything into Cartesian coordinates and then use the kinematics equations in the x and y dimensions independently. Once you have final answers, you can convert them back into polar coordinates.