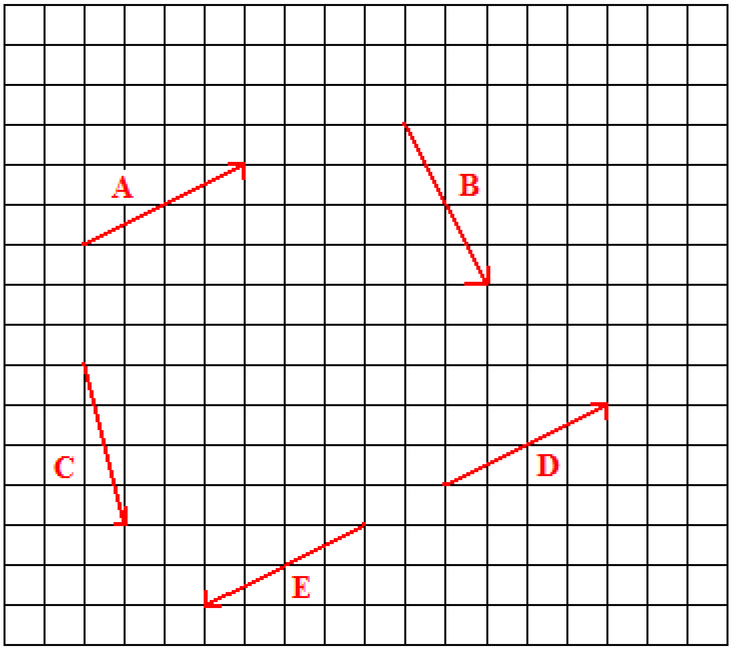
The dot product and work



We've seen already that vectors can be added and subtracted. There are also two useful ways vectors can be multiplied. The first of these is called the vector dot product, or just dot product.

Suppose we wanted to take the vector dot product of **A** and **C**. It would be written **A▪C**. The heavy dot between the two indicated the dot product.

It can be computed with arithmetic: **A▪C =** $A_x \cdot C_x + A_y \cdot C_y = (4 \cdot 1) + (2 \cdot -4) = -4$. The answer is just a number without direction, which is a scalar, so the dot product can also be called the scalar product.

All you are doing is multiplying the two x-components, multiplying the two y-components, and then adding those two products together. Try it with **A▪B** and you should get an answer of zero.

Conceptually, the dot product is a measure of how parallel two vectors are to each other. Vectors **A** and **B** are perpendicular, so they are not at all parallel, making the dot product zero.

You can also see that **A▪C = C▪A** because $A_x \cdot C_x + A_y \cdot C_y = C_x \cdot A_x + C_y \cdot A_y$

If you take the dot product **D▪E** or **E▪D**, you will get a high negative number, -20, because the two vectors are highly anti-parallel.

Answer Webassign Question 1

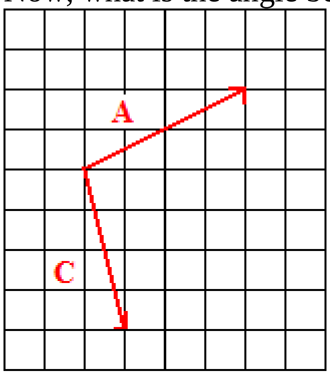
Another way of computing the dot product, useful in other contexts, is to use the cosine function. Let's use **A▪C** again.

Here, **A▪C** = |A|·|C|·cosθ

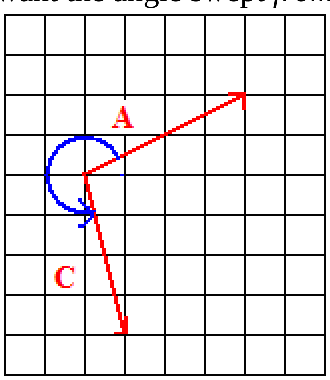
This is read as A-dot-C equals the magnitude of A times the magnitude of C times cosine of the angle between the two vectors.

The magnitude of A we can get from the Pythagorean Theorem, |A| = $\sqrt{4^2 + 2^2}$ = $\sqrt{20}$
Likewise, |C| = $\sqrt{17}$

Now, what is the angle between the two vectors? Let me draw them with a common origin.



If you use the inverse tangent function for vector **A** where θ = $\tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{2}{4})$, you will find it has an angle of 26.565º and **C** has an angle of $\tan^{-1}(\frac{-4}{1})$ = 284º. The angle used in the dot product is the angle *from* the first vector, counter-clockwise, *to* the second vector. Because we are working with **A▪C**, we want the angle swept *from* **A** *to* **C**.



This angle is 284 – 25.565 = 257.435º, making the full dot product:

A▪C =$\sqrt{20}$ ·$\sqrt{17}$·cos(257.435º) = -4.

This is the same result as before. It is true that if you swept out the angle clockwise, you would get the same answer. It doesn't matter for the dot product because the cosine function is symmetric about the y-axis. But it will matter later for the vector cross product, so the counter-clockwise sweep is a good habit to begin.

Answer Webassign Question 2

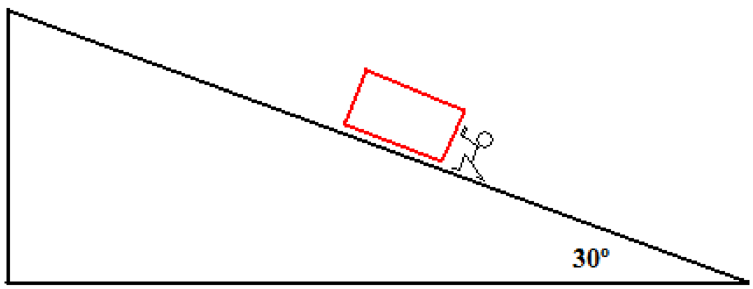
How these two definitions of the dot product are the same, I will leave for an appendix at the end.

In physics, the concept of *work* is defined as the dot product of force and displacement.

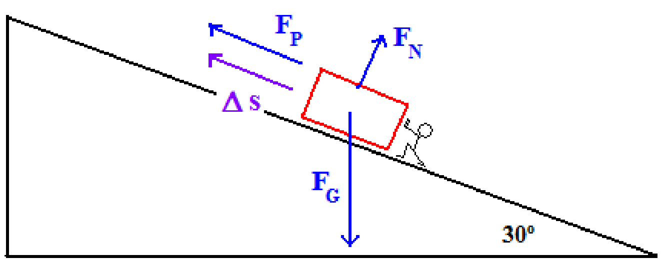
$W = \mathbf{F} \cdot \mathbf{\Delta s}$ or, using the second method described above, $W = |F| \cdot |\Delta s| \cdot \cos\theta$

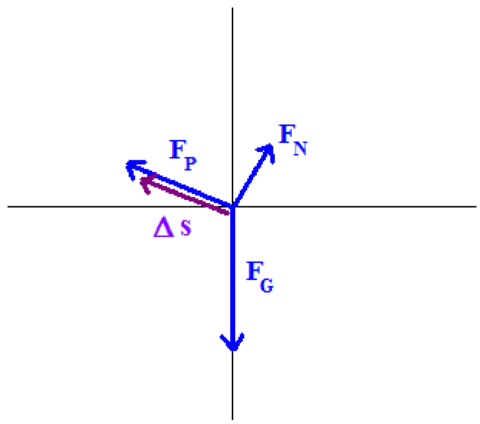
Work is the magnitude of the force times the magnitude of the displacement times the cosine of the angle between them. Again, any dot product is a scalar, so the physical concept of work is a scalar.

Let's see how this can be used in an example. Suppose you push a 20kg crate up a 30º frictionless incline by applying a force of 150N directly up the incline over a distance of 4m.



Now, I'll redraw the situation with all of the relevant vectors and then a vector diagram off to the side.





Because there are three forces during the displacement, there are three works to calculate:

$W_P = |F_P| \cdot |\Delta s| \cdot \cos\theta$

$W_G = |F_G| \cdot |\Delta s| \cdot \cos\theta$

$W_N = |F_N| \cdot |\Delta s| \cdot \cos\theta$

Although they all have a common displacement, the forces and angles will be different for each. The force of the push is given as 150N, the force of gravity is 196N (again, magnitudes only), and the normal force on an incline is $m \cdot g \cdot \cos(\theta_{incline}) = (20)(9.8)(\cos 30º) = 170N$.

For each, the angle is from the force vector, counter-clockwise, to the displacement vector according to the vector diagram above. For the push, the angle between the push force and the displacement is 0º. For the force of gravity, the angle from the force of gravity to the displacement vector is 240º. And for the normal force, this angle is 90º.

Putting all six of those numbers into the equations above yields:

$W_P = (150N)(4m)(\cos 0^\circ) = 600N\cdot m$

$W_G = (196N)(4m)(\cos 240^\circ) = -392N\cdot m$

$W_N = (170N)(4m)(\cos 90^\circ) = 0N\cdot m$

Because work is a scalar, we can simply add up all of these numbers to calculate the net work:

$\Sigma W = 208N\cdot m$

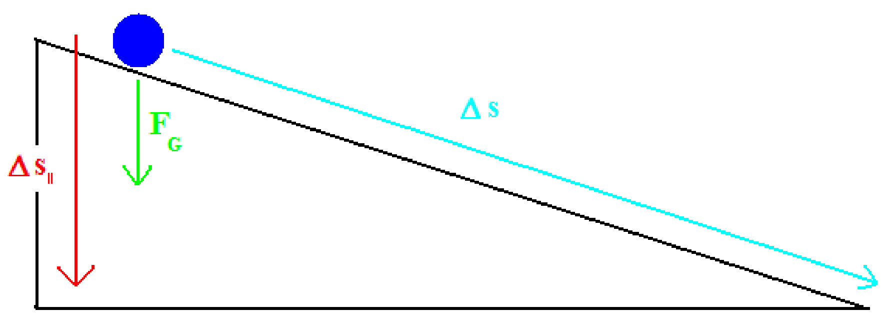
This can also be written as 208J, as 1 Joule = 1 Newton-meter.

Answer Webassign Question 3

Another way of conceptualizing work is as the force applied over the displacement parallel to that force.

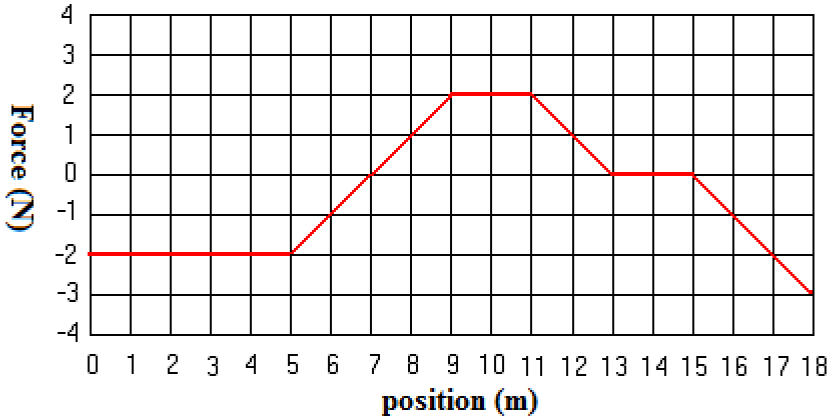
$W = F\cdot\Delta s_{||}$ where || means *parallel to*

For instance, for a ball rolling down an incline, the force of gravity is straight down, so the displacement parallel to that force is the vertical displacement.

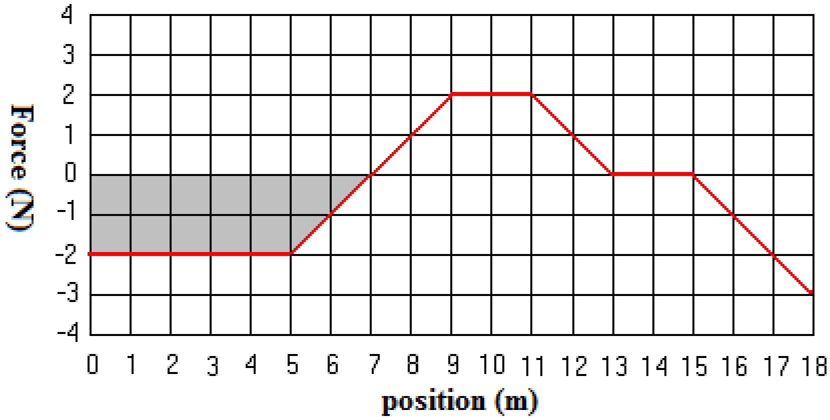


Answer Webassign Question 4

Restricting ourselves to one-dimension, if there was a graph of force versus position, the area under that curve would equal the work because the area is the product of force and displacement.



For instance, the work from 0s to 7m would be the area trapped to the x-axis, which is -12N·m or -12J.



We can also relate this concept of work to Newton's second law.

$\Sigma W = \Sigma F \cdot \Delta s = (m \cdot a) \cdot \Delta s$

Rearranging the v_f^2 equation produces $\frac{v_f^2 - v_i^2}{2} = a \cdot \Delta s$, therefore

$\Sigma W = m \cdot \frac{v_f^2 - v_i^2}{2} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_f^2 = K_f - K_i = \Delta K$

where we define the concept of kinetic energy as $K = \frac{1}{2} m v^2$.

This idea of $\Sigma W = \Delta K$ is called the work-energy theorem, even though it's not really an energy equation, but rather a disguised form of Newton's second law of motion.

Answer Webassign Question 5

To review:

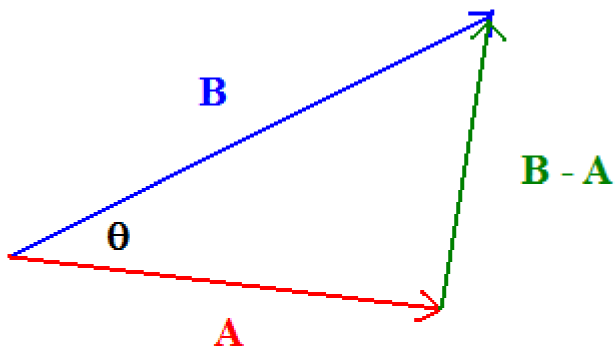
$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = |A|\cdot|B|\cdot\cos\theta$

$W = \mathbf{F}\cdot\mathbf{\Delta s}$

The area under a force/position function is the work input.

$\Sigma W = \Delta K$ where $K = \frac{1}{2}m\cdot v^2$

Appendix – proof of the equivalence of the dot product definitions



If you look at the vector diagram above, **A** + (**B** – **A**) = **B** is a valid relationship

The law of cosines states $c^2 = a^2 + b^2 + 2|a||c|\cos\theta$

Here, that would be:

$(\mathbf{B} - \mathbf{A})^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2|A||B|\cos\theta$

Expanding the left side yields

$\mathbf{B}^2 - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{A}^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2|A||B|\cos\theta$

Something like $\mathbf{A}\cdot\mathbf{A} = |A||A|\cos(0^\circ) = A^2$, therefore

$B^2 - 2\mathbf{A}\cdot\mathbf{B} + A^2 = A^2 + B^2 + 2|A||B|\cos\theta$

So $\mathbf{A}\cdot\mathbf{B} = |A||B|\cos\theta$