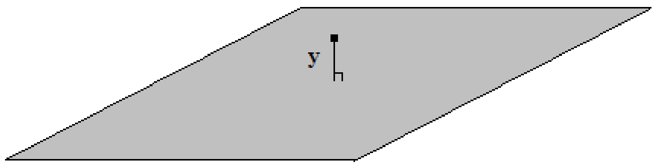
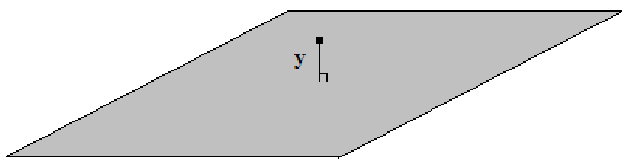
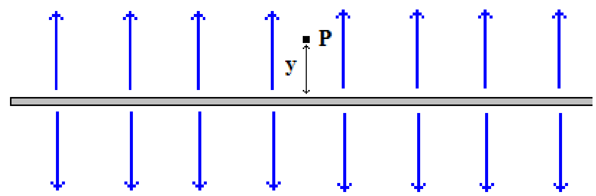
Capacitors

Suppose we have a very large, flat sheet of metal with a charge density (measured in $\frac{C}{m^2}$) of +σ. We want to find the electric field at a point, *P*, some height, *y*, above the sheet. We will take *y* to be small enough that the sheet is essentially infinite in expanse.

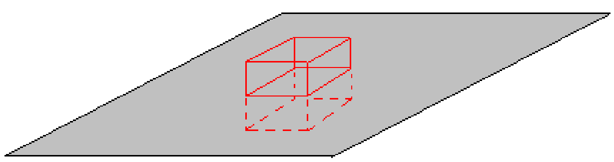


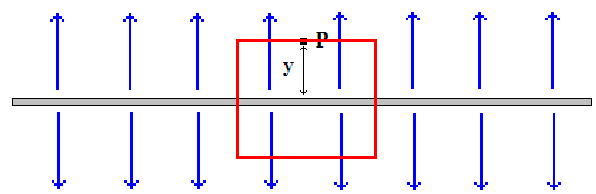
If we look at a side view of the sheet, we will have electric field lines pointing away from the sheet both upwards and downwards.





We can then imagine enclosing a portion of this sheet in a Gaussian box where one of the sides contains the point *P*.





Let's say the box is a cube with sides of *2y*.

From Gauss's law, $Q_{enclosed} = \varepsilon_0 \cdot \oiint E \cdot dA$

$Q_{enclosed}$ in the box is (2y)·(2y)·σ

The electric field is passing through the top and bottom sides of the box, so

$\oiint E \cdot dA = E \cdot [2 \cdot (2y) \cdot (2y)]$

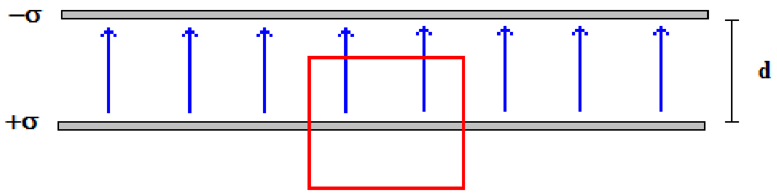
$(2y) \cdot (2y) \cdot \sigma = \varepsilon_0 \cdot E \cdot [2 \cdot (2y) \cdot (2y)]$

$$E = \frac{\sigma}{2 \cdot \varepsilon_0}$$

The electric field is actually independent of the height, *y*.

Answer Webassign Question 1

A very similar situation is if we have one large flat sheet with a charge density +*σ* parallel to another large flat sheet with a charge density –*σ*.



If we now just enclose a portion of the bottom plate with a box of sides *x*, we have:

$Q_{enclosed} = \varepsilon_0 \cdot \oiint E \cdot dA$

$Q_{enclosed} = \sigma \cdot x^2$

$\oiint E \cdot dA = E \cdot x^2$

$\sigma \cdot x^2 = \varepsilon_0 . E \cdot x^2$

$$E = \frac{\sigma}{\varepsilon_0}$$

If the distance between the plates is *d*, it's fairly easy to determine the voltage between the plates.

$$\Delta V = -\int \boldsymbol{E \cdot dR} = -\left(\frac{\sigma}{\varepsilon_0}\right) \cdot d$$

A magnitude of $\Delta V = \frac{\sigma \cdot d}{\varepsilon_0}$

This system is often called a *parallel plate capacitor* where its capacitance, *C*, is defined as:

$C = \frac{Q}{\Delta V}$ measured in Farads where $1F = 1\,\frac{C}{V}$

Capacitance is the magnitude of the charge stored on either plate divided by the magnitude of the voltage across the plates.

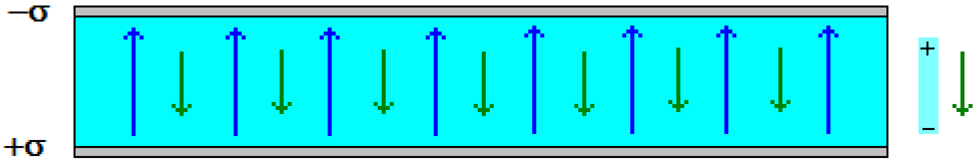
If each plate has an area, *A*, then $Q = \sigma \cdot A$ and

$C = \frac{\sigma \cdot A}{\frac{\sigma \cdot d}{\varepsilon_0}}$ or $C = \frac{A \cdot \varepsilon_0}{d}$ for a parallel plate capacitor

Answer Webassign Question 2

Suppose we now take the two plates and fill the space between them with some material, called a *dielectric*, which is polarized, to a greater or lesser extent, by the two charged plates.

The molecules will be polarized so that the negative side of the dipole is pulled towards the positive plate and the positive side of the dipole is pulled towards the negative plate. This produces an electric field from the molecules which is in the opposite direction of the plates' field.



Now the net electric field between the plates is the difference between the magnitudes of E_{plates} and $E_{dielectric}$.

$E_{new} = E_{plates} - E_{dielectric}$

The ratio of the original field to this weakened field is the dielectric constant of the material, κ.

$$\kappa = \frac{\text{Eplates}}{E_{plates} - E_{dielectric}}$$

κ is greater than one and increases as the material increases in ease of polarization

Now the electric field (E_{new}) between the plates is $\frac{E_{plates}}{\kappa}$

$|\Delta V| = \int \boldsymbol{E \cdot dR} = \left(\frac{E_{plates}}{\kappa}\right)\cdot d$ and $Q = \sigma\cdot A$

$$C = \frac{Q}{\Delta V} = \frac{\sigma\cdot A}{\left(\frac{E_{plates}}{\kappa}\right)\cdot d}$$

Where $E_{plates} = \frac{\sigma}{\varepsilon_0}$ from the beginning

All simplifies to:

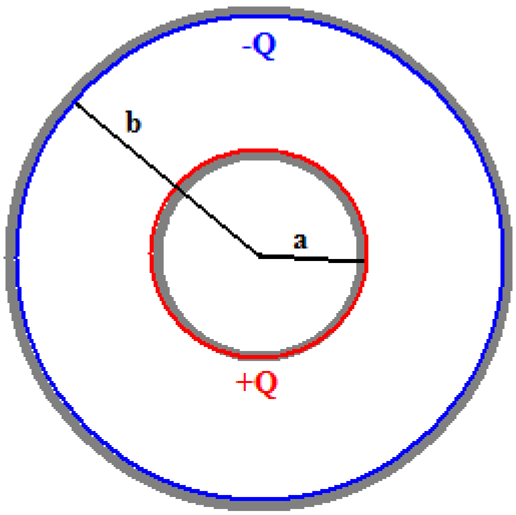
$$C = \frac{\kappa\cdot A\cdot\varepsilon_0}{d}$$ for a parallel plate capacitor with a dielectric

Because κ is always greater than one, a dielectric increases the capacitance of a capacitor.

Answer Webassign Question 3

Two more general capacitors are spherical capacitors and cylindrical capacitors.

A spherical capacitor has two shells, one of radius *a* and one of radius *b*. The inner shell may carry a charge +*Q* while the outer shell carries a charge –*Q*.



We still use the definition for capacitance, $C = \frac{Q}{\Delta V}$

We just need to find the change in potential between the outer and inner spheres.

As always, $\Delta V = -\int \boldsymbol{E} \cdot \boldsymbol{dR}$ where $E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R^2}$ from Gauss's law

$$\Delta V = -\frac{Q}{4\pi\varepsilon_0} \left|\int_b^a \frac{1}{R^2} \cdot dR\right| \cdot \cos 180^\circ$$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0} \cdot \left[\frac{1}{a} - \frac{1}{b}\right]$$

Putting this into $C = \frac{Q}{\Delta V}$ yields

$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$ or $$C = \frac{4\pi\varepsilon_0 \cdot ab}{(b-a)}$$ for a spherical capacitor

Answer Webassign Question 4

The cylindrical capacitor derivation takes the exact same form, but the equation for the electric field is different when between cylinders. As derived in the notes on linear charges:

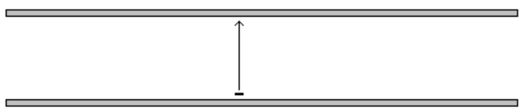
$$E = \frac{1}{2\pi \cdot \varepsilon_0} \cdot \frac{\lambda}{R}$$

$$\Delta V = - \frac{\lambda}{2\pi\varepsilon_0} \left|\int_b^a \frac{1}{R} \cdot dR\right| \cdot \cos 180^o$$

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0} \cdot \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{\lambda \cdot L}{\frac{\lambda}{2\pi\varepsilon_0} \cdot \ln\left(\frac{b}{a}\right)} = \frac{2\pi \cdot \varepsilon_0 \cdot L}{\ln\left(\frac{b}{a}\right)}$$

$C = \dfrac{2\pi \cdot \varepsilon_0 \cdot L}{\ln\left(\frac{b}{a}\right)}$ for a cylindrical capacitor of length *L* and radii *a* and *b*



Suppose we begin with two neutral, metal, parallel plates. If we pull an electron from the bottom plate and place it on the top plate, it will require an input of energy. This is because the bottom plate will be left with a positive charge and pull downward on the electron as we lift it upwards.

Once the electron has been placed on the top plate, we can drag another electron from the bottom to the top, but this will take even more energy as the bottom plate pulls down harder and the top plate is also now pushing down on the mobile electron.

Mathematically, the energy gained by the capacitor looks like this: $dU = \Delta V \cdot dq$

A little bit of energy (dU) arises in the capacitor when a little bit of charge (dq) is pulled from the bottom to the top plate across the capacitor's voltage (ΔV). The total energy stored in the capacitor is then the sum of all of these dU values:

$U_{final} = \int dU = \int \Delta V \cdot dq = \int \frac{q}{C} dq$ because $\Delta V = \frac{Q}{C}$ by the definition of capacitance

If the boundaries for charge are zero and U_{final}, then the integral becomes $U_{final} = \frac{q^2}{2C}$.

Answer Webassign Question 5

And if $C = \frac{Q}{\Delta V}$, the energy in a capacitor can be written in three forms:

$$\mathbf{U = \frac{Q^2}{2C}} \qquad \mathbf{U = \frac{1}{2}C \cdot \Delta V^2} \qquad \mathbf{U = \frac{1}{2}Q \cdot \Delta V}$$