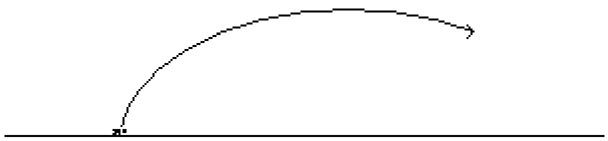
Parametric Equations



Suppose a cricket jumps off of the ground with an initial velocity v_0 at an angle θ. If we take his initial position as the origin, his horizontal and vertical positions follow the equations:

$x = v_0 \cdot \cos\theta \cdot t$

$y = v_0 \cdot \sin\theta \cdot t - \frac{g \cdot t^2}{2}$

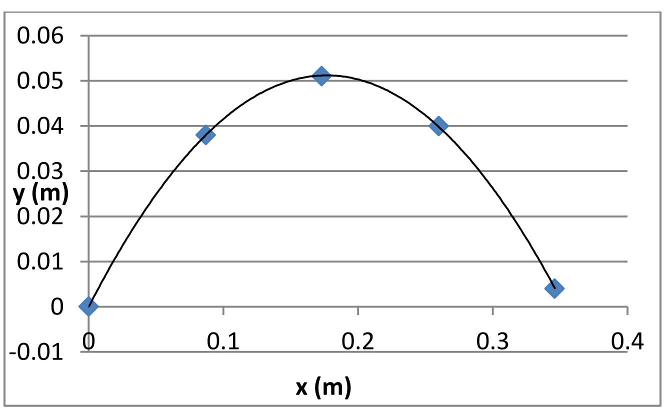
These are called *parametric* equations, with time being the parameter upon which the positions depend.

If we had numeric values for the initial velocity and angle, it would be easy to make a chart of the motion. Supposing v_0 = 2m/s and θ = 30º.

$x = 1.73 \cdot t$
$y = t - 4.9 \cdot t^2$

Time (s)	x (m)	y (m)
0.0	0	0
0.05	0.087	0.038
0.10	0.173	0.051
0.15	0.260	0.040
0.20	0.346	0.004

The graph of which is a simple parabola.



We can also eliminate the parameter through algebraic substitution:

$x = v_0 \cdot \cos\theta \cdot t$ so $t = \frac{x}{v_0 \cdot \cos\theta}$

$y = v_0 \cdot \sin\theta \cdot t - \frac{g \cdot t^2}{2}$ so $y = v_0 \cdot \sin\theta \cdot (\frac{x}{v_0 \cdot \cos\theta}) - \frac{g}{2} \cdot (\frac{x}{v_0 \cdot \cos\theta})^2$

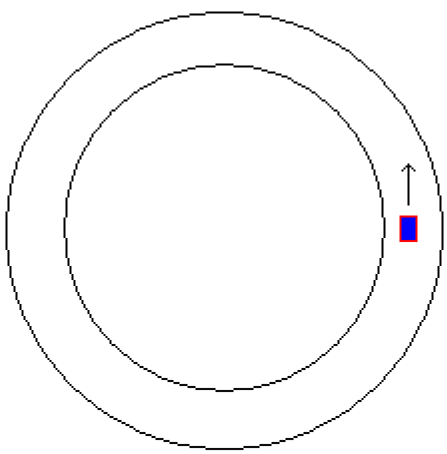
$y = x \cdot \tan\theta - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2\theta}$ which is exactly the parabola graphed above

So the question arises, if we can extract the parameter and simply have y as a function of x, what is the value of parametric equations in the first place? There are several answers:

1. It is not always possible to extract the parameter from simultaneous equations. Often, even when possible, the result is uninformatively complex.

2. In many physical situations, we would *like* to know how the positions depend upon the parameter. For the cricket, the parametric equations tell us where he is at any given time.

3. The parametric equations provide a direction for the curve. In our example, the cricket was clearly jumping from left to right, but extracting time from the equations and writing y as a function of x removed this information.



We can see those advantages with another example, a car driving around in circle, counter-clockwise. Suppose the speed is a constant *v*, the radius of the track is *R*, the origin is at the center of the track, and the car has the position shown at time zero. The parametric equations for the car's horizontal and vertical positions are then:

$x = R \cdot \cos(\frac{v}{R} \cdot t)$

$y = R \cdot \sin(\frac{v}{R} \cdot t)$

We can extract the parameter of time as follows:

$x^2 + y^2 = R^2 \cdot \cos^2\left(\frac{v}{R} \cdot t\right) + R^2 \cdot \sin^2\left(\frac{v}{R} \cdot t\right) = R^2$ or just $x^2 + y^2 = R^2$

But, again, this Cartesian version has the following disadvantages:

1. $x^2 + y^2 = R^2$ is not a function, so cannot be written in simple functional notation

2. $x^2 + y^2 = R^2$ does not tell us where the car is at any given time

3. $x^2 + y^2 = R^2$ does not tell us that the car is moving counter-clockwise

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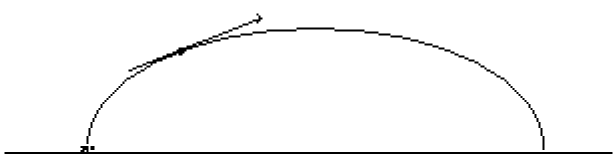
What calculus can be applied to parametric equations?



Suppose we wanted the instantaneous trajectory of the cricket at any time or position, that is, we want to find the tangent to the spatial curve, or $\frac{dy}{dx}$.

One way is to simply differentiate the equation $y = x \cdot \tan\theta - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2\theta}$ and find

$\frac{dy}{dx} = \tan\theta - \frac{g \cdot x}{v_0^2 \cdot \cos^2\theta}$

Another is to use the parametric equations. Let's begin with the chain rule:

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

If $x = v_0 \cdot \cos\theta \cdot t$ then $\frac{dx}{dt} = v_0 \cdot \cos\theta$

If $y = v_0 \cdot \sin\theta \cdot t - \frac{g \cdot t^2}{2}$ then $\frac{dy}{dt} = v_0 \cdot \sin\theta - g \cdot t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_0 \cdot \sin\theta - g \cdot t}{v_0 \cdot \cos\theta}$ or $\frac{dy}{dx} = \tan\theta - \frac{g \cdot t}{v_0 \cdot \cos\theta}$ as a function of time

We can also replace t with $\frac{x}{v_0 \cdot \cos\theta}$ from earlier to get

$\frac{dy}{dx}$ = tanθ - $\frac{g \cdot x}{v_0^2 \cdot \cos^2\theta}$ as we found before

Incidentally, it's a good habit, once equations are derived, to play with various values. Do the equations make sense when t = 0? Do they make sense where $\frac{dy}{dx}$ = 0? Why is t in the numerator and cosθ in the denominator?

Now suppose we want the arc length traveled by the cricket. Let's define ds to be an infinitely small piece of this arc length. Then $ds^2 = dx^2 + dy^2$

By the chain rule, dy = $\frac{dy}{dx} \cdot dx$, so $ds^2 = dx^2 + (\frac{dy}{dx} \cdot dx)^2 = [1 + (\frac{dy}{dx})^2] \cdot dx^2$

Taking the square root of both sides and integrating yields:

Arc length = $\int \sqrt{1 + (\frac{dy}{dx})^2} \cdot dx$

By symmetry, we could also use

Arc length = $\int \sqrt{1 + (\frac{dx}{dy})^2} \cdot dy$

If we use the first form for our cricket, we would have

Arc length = $\int \sqrt{1 + (\tan\theta - \frac{g \cdot x}{v_0^2 \cdot \cos^2\theta})^2} \cdot dx$

where the boundaries for the full flight would be x = 0 to x = $\frac{v_0^2 \cdot \sin(2\theta)}{g}$ (from the range equation)

That is not a particularly pleasant integral, so let's try using parametric equations:

If we remember $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ then arc length = $\int \sqrt{1 + (\frac{dy}{dx})^2} \cdot dx$ becomes:

Arc length = $\int \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \cdot \frac{dx}{dt} \cdot dt$ or arc length = $\int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$

For our cricket, arc length = $\int \sqrt{(v_0 \cdot \cos\theta)^2 + (v_0 \cdot \sin\theta - g \cdot t)^2} \cdot dt$

where the boundaries for the full flight are t = 0 and t = $\frac{2v_0 sin\theta}{g}$

This is still a complex integral and it takes some work, but the solution is:

Arc length = $\frac{v_0^2 cos^2\theta}{2g} \cdot [\, 2\sec(\theta) \cdot \tan(\theta) + \ln\left|\frac{1+\sin(\theta)}{1-\sin(\theta)}\right| \,]$

Lastly, what is the area under the arc jumped by the cricket?

We could simply integrate $y = x \cdot \tan\theta - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot cos^2\theta}$ between the boundaries of x = 0 and $x = \frac{2v_0^2 \cdot \sin(\theta)\cos(\theta)}{g}$

This would be:

Area = $\int x \cdot \tan\theta - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot cos^2\theta} \cdot dx = \frac{x^2}{2} \cdot \tan\theta - \frac{g \cdot x^3}{6 \cdot v_0^2 \cdot cos^2\theta}$

And with the boundaries entered: Area = $\frac{2v_0^4 sin^3\theta \cdot cos\theta}{3g^2}$

What would this look like using parametric equations?

Area = $\int_{x_i}^{x_f} y(x) \cdot dx = \int_{t_i}^{t_f} y(t) \cdot \left(\frac{dx}{dt}\right) \cdot dt$ by using substitution for definite integrals

For our cricket, $y(t) = v_0 \cdot \sin\theta \cdot t - \frac{g \cdot t^2}{2}$ and $\frac{dx}{dt} = v_0 \cdot \cos\theta$

With the boundaries of t = 0 and t = $\frac{2v_0 sin\theta}{g}$

Area = $\int_0^{\frac{2v_0 sin\theta}{g}} \left(v_0 \cdot \sin\theta \cdot t - \frac{g \cdot t^2}{2}\right) \cdot (v_0 \cdot \cos\theta) \cdot dt$

Area = $\frac{2v_0^4 sin^3\theta \cdot cos\theta}{3g^2}$ as we found before