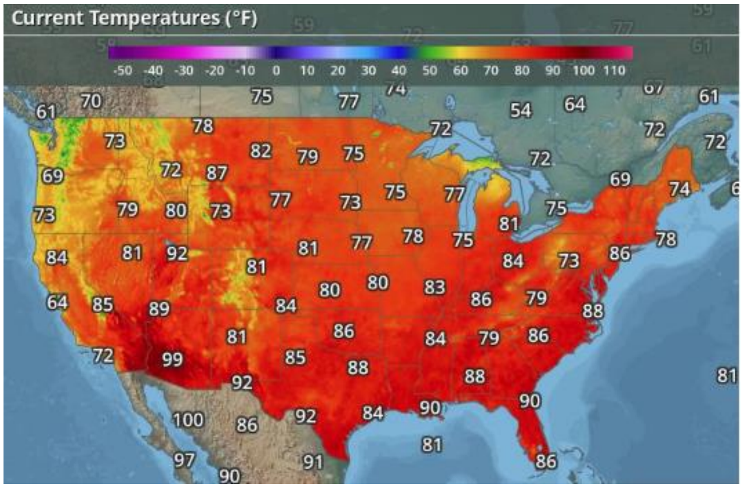
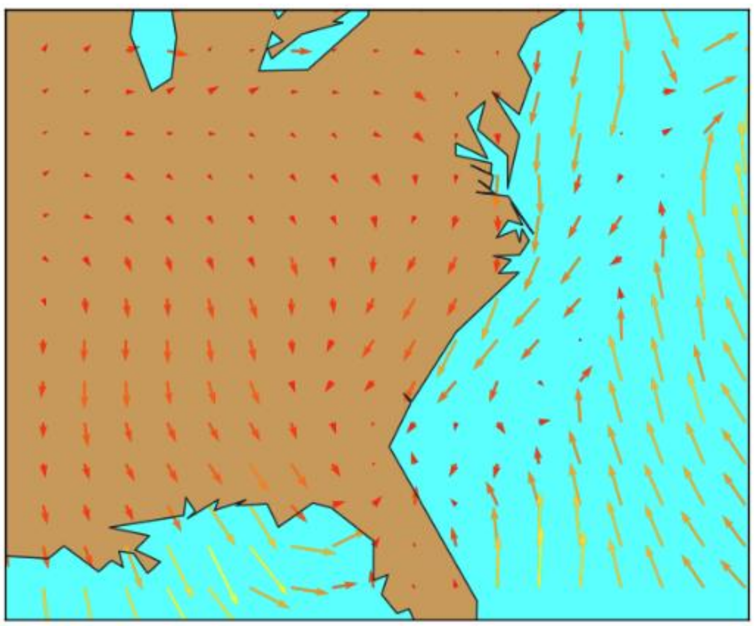
Vector Fields



The map shown above is an example of a *scalar field*, where the range is a set of points in two-dimensional space and the domain is a set of temperatures associated with those points at some instant in time. The field is scalar because temperature is a scalar quantity, one with magnitude but without direction.



This map, however, is a *vector field*, where the range is also a set of points in two-dimensional space, but the domain is a set of vectors associated with those points. Here, those vectors represent wind velocity at some instant in time. Diagrammatically, the tail of the vector is placed at the spatial point where the wind velocity is measured; the length of the vector represents the magnitude of that wind velocity.

If we let $\vec{F}$ represent some general field, then (in two dimensions):

$\vec{F}$ (x,y) = P(x,y)·$\hat{\boldsymbol{\imath}}$ + Q(x,y)·$\hat{\boldsymbol{\jmath}}$ where P is the x-component of the field as a function of position in the x-y plane and Q is the y-component of the field as a function in the x-y plane. As always, i-hat is the x-dimension unit vector and j-hat is the y-dimension unit vector.

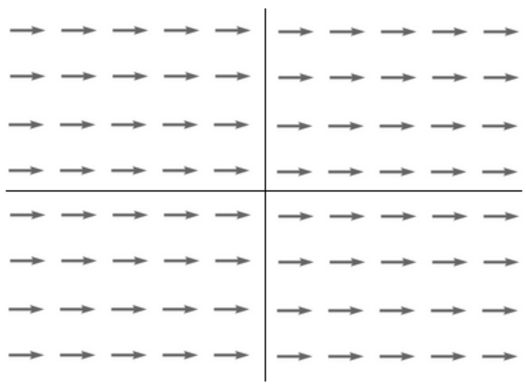
Vector fields in three dimensions can be difficult to represent and visualize, but, mathematically, only require the obvious extension:

$\vec{F}$ (x,y) = P(x,y,z)·$\hat{\boldsymbol{\imath}}$ + Q(x,y,z)·$\hat{\boldsymbol{\jmath}}$ + R(x,y,z)·$\hat{\boldsymbol{k}}$

The first skill to develop is drawing or visualizing a given vector field function. Take an easy case like

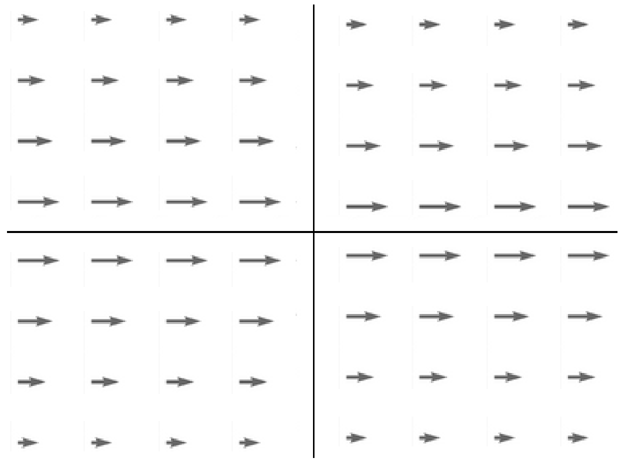
$\vec{F}$ (x,y) = 5$\hat{\boldsymbol{\imath}}$

Here, the vector field always has a magnitude of five and is directed to the right:



Obviously, not every point in space can be represented by an arrow; that would create an uninterpretable diagram of pure black.

What about a function like $\vec{F}$ (x,y) = (25 – 5y^2)$\hat{\boldsymbol{\imath}}$ where -5 > y > 5.



Here you have vectors always pointed to the right with magnitudes of 25 along the x-axis which gradually drop to zero at y = 5 and y = -5. This function might model laminar fluid flow along some channel with a width of ten vertical units. The flow velocity is maximum in the middle of the channel and drops to zero at the borders to satisfy the boundary conditions of the channel.

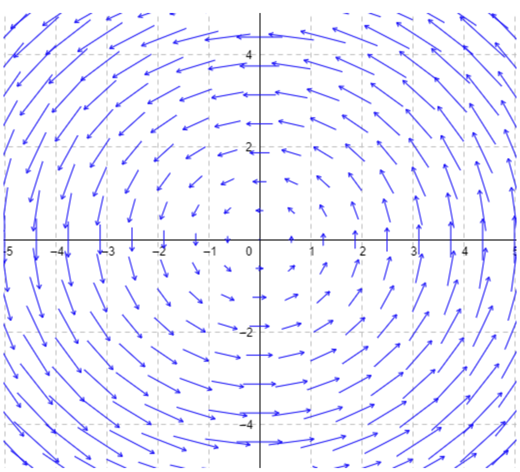
As these functions become more complex and visualization becomes less straightforward, it can be useful to have some method to assist the process of translating a function into a vector field map.

For example, what does $\vec{F}$ (x,y) = – 5y·$\boldsymbol{\hat{\imath}}$ + 5x·$\boldsymbol{\hat{\jmath}}$ look like?

First, make a table with some convenient spatial coordinates. Then complete the table to get a sense of what arrows to draw and how long those arrows are and in what direction they point. Here is such a table for the function above:

Spatial coordinates:	Field vector at those coordinates:	What to draw:
(x,y) = (0,0)	$\vec{F}$ (x,y) = 0·$\boldsymbol{\hat{\imath}}$ + 0·$\boldsymbol{\hat{\jmath}}$	At the origin, no vector exists
(x,y) = (1,0)	$\vec{F}$ (x,y) = 5·$\boldsymbol{\hat{\jmath}}$	At position (1,0), a vector of 5 upwards.
(x,y) = (2,0)	$\vec{F}$ (x,y) = 10·$\boldsymbol{\hat{\jmath}}$	At position (2,0), a vector of 10 upwards
(x,y) = (0,1)	$\vec{F}$ (x,y) = -5·$\boldsymbol{\hat{\imath}}$	At position (0, 1) a vector of 5 leftwards.
(x,y) = (0,2)	$\vec{F}$ (x,y) = -10·$\boldsymbol{\hat{\imath}}$	At position (0,2), a vector of 10 leftwards.
(x,y) = (-1,0)	$\vec{F}$ (x,y) = -5·$\boldsymbol{\hat{\jmath}}$	At position (-1,0), a vector of five downwards.
(x,y) = (-2,0)	$\vec{F}$ (x,y) = -10·$\boldsymbol{\hat{\jmath}}$	At position (-2,0), a vector of ten downwards.
(x,y) = (0,-1)	$\vec{F}$ (x,y) = 5·$\boldsymbol{\hat{\imath}}$	At position (0,-1), a vector of five rightwards.
(x,y) = (0,-2)	$\vec{F}$ (x,y) = 10·$\boldsymbol{\hat{\imath}}$	At position (0,-2), a vector of ten rightwards.

These vectors agree with field diagram generated by an online vector field plotter for the function given above:

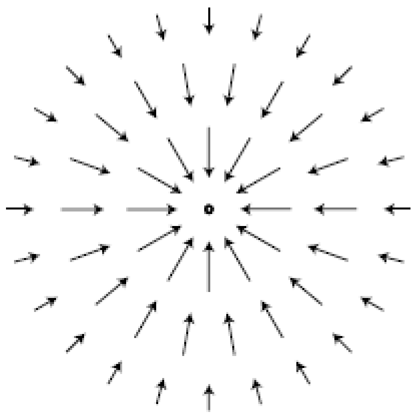


The function itself might represent the tangential velocities of points on a disk spinning counter-clockwise with an angular velocity of 5 rad/s.

Fields you may be well-familiar with include the gravitational field of a point mass (or some object idealized as a point mass) and the electric field of a point charge.

The magnitude of a point mass's gravitational field follows the equation:

g = $\frac{Gm}{R^2}$ while the direction is always radially inward towards the point mass, producing (in two-dimensions) the vector field diagram below:



To notate both the magnitude and direction of the field simultaneously, we can write it as:

$\vec{g}$ = - $\frac{Gm}{R^2} \cdot \hat{R}$ where $\hat{R}$ is a unit vector pointing radially outward from the origin

This unit vector does not affect the magnitude of the field, but when coupled with the leading negative sign indicates that the field is always towards the origin, antiparallel to the outgoing unit radial vector itself.

Another way to notate the same function is $\vec{g}$ = - $\frac{Gm}{R^3} \cdot \vec{R}$

where $\vec{R}$ is the radial vector (magnitude and direction) of where the field is being measured. The magnitude then simplifies algebraically to $\frac{Gm}{R^2}$ and the combination of the leading negative and radial vector account for the radially-inward gravitational field.

Likewise, the electric field can be written as:

$\vec{E} = \frac{kQ}{R^2} \cdot \hat{R}$ or $\vec{E} = \frac{kQ}{R^3} \cdot \vec{R}$

radially outward when Q is positive and radially inward (like gravity) when Q is negative.

We can also write either field in components. Let's take the gravitational field:

$\vec{g} = \vec{g}_x + \vec{g}_y + \vec{g}_z$

$= -g \cdot \cos\theta_x \cdot \hat{\boldsymbol{\imath}} + -g \cdot \cos\theta_y \cdot \hat{\boldsymbol{\jmath}} + -g \cdot \cos\theta_z \cdot \hat{\boldsymbol{k}}$ using directional cosines

$= -g \cdot \frac{x}{R} \cdot \hat{\boldsymbol{\imath}} + -g \cdot \frac{y}{R} \cdot \hat{\boldsymbol{\jmath}} + -g \cdot \frac{z}{R} \cdot \hat{\boldsymbol{k}}$

$= \frac{-Gm}{R^2} \left(\frac{x}{R} \cdot \hat{\boldsymbol{\imath}} + \frac{y}{R} \cdot \hat{\boldsymbol{\jmath}} + \frac{z}{R} \cdot \hat{\boldsymbol{k}}\right)$

$= \frac{-Gmx}{R^3} \cdot \hat{\boldsymbol{\imath}} + \frac{-Gmy}{R^3} \cdot \hat{\boldsymbol{\jmath}} + \frac{-Gmz}{R^3} \cdot \hat{\boldsymbol{k}}$

You may recall in unit 14 that if there is some scalar function, f(x,y), for instance, altitude on a topographic map, then there is also a vector function called the gradient such that

$$\nabla f(x,y) = f_x(x,y)\hat{\boldsymbol{\imath}} + f_y(x,y)\hat{\boldsymbol{\jmath}}$$

For instance, if f(x,y) = $6xy^2$

$\nabla f(x,y) = 6y^2 \cdot \hat{\boldsymbol{\imath}} + 12xy \cdot \hat{\boldsymbol{\jmath}}$

This function can be considered a vector field and is thus called the gradient vector field.

As usual, extending into three dimensions is straightforward:

$$\nabla f(x,y,z) = f_x(x,y,z)\hat{\boldsymbol{\imath}} + f_y(x,y,z)\hat{\boldsymbol{\jmath}} + f_z(x,y,z)\hat{\boldsymbol{k}}$$

This combination of a scalar field and the gradient of that scalar field being a vector field is very useful in many physical situations, for example, in electrostatics. If we have some scalar field of electric potential, V(x,y,z), then the gradient of that scalar field (times -1) is the vector field, called simply the electric field. This is written very succinctly as:

$$\vec{E} = -\nabla V$$

Which you way have seen before as the similar:

$$\vec{E} = -\frac{dV}{dR}$$

As discussed in unit 14, the gradient vector points towards increasing scalar values, has a high magnitude when those values change rapidly over space, and points perpendicular to level curves or surfaces (in electrostatics, called equipotential lines or equipotential surfaces).

Vector fields that can be based upon scalar fields in this way are said to be *conservative*. We can see this is true for the electric field if we define electric potential as:

$$V(x,y,z) = \frac{kQ}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore,

$\nabla V(x, y, z) =$

$V_x(x, y, z)\hat{\boldsymbol{\imath}} + V_y(x, y, z)\hat{\boldsymbol{\jmath}} + V_z(x, y, z)\hat{\boldsymbol{k}} =$

$\frac{-kQx \cdot \hat{\boldsymbol{\imath}}}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-kQy \cdot \hat{\boldsymbol{\jmath}}}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-kQz \cdot \hat{\boldsymbol{k}}}{(x^2 + y^2 + z^2)^{3/2}} =$

$= \frac{-kQ}{R^3}(x \cdot \hat{\boldsymbol{\imath}} + y \cdot \hat{\boldsymbol{\jmath}} + z \cdot \hat{\boldsymbol{k}})$

which, as shown earlier, is the component form of $-\vec{E}$.