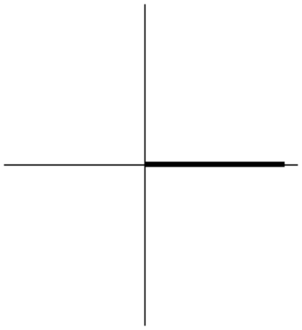
Line integrals



In the simple diagram above, we have a bar with length L sitting along the positive x-axis. Suppose the bar has a mass density that depends upon x such that:

$\lambda = \alpha \cdot x^2$

A little piece of the bar therefore has a little mass:

$dm = \lambda \cdot dx = \alpha \cdot x^2 \cdot dx$

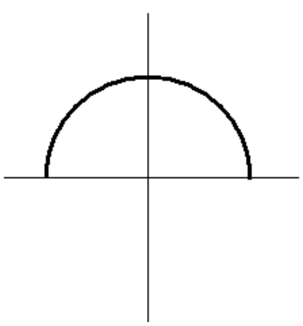
And the entire bar therefore has a mass:

$M = \int_0^L \alpha \cdot x^2 \cdot dx$

$M = \frac{\alpha \cdot L^3}{3}$

This is a very simple example of a line integral. For a more complex example, suppose we have a wire bent into a semicircular arc with a radius of 2 where the linear mass density of the wire depends upon the position in the x-y plane such that :

$\lambda = x^2 \cdot y$



Here, instead of imagining taking steps, *dx*, along the x-axis, we imagine taking steps, *ds*, along the curve itself so that:

$M = \int \lambda \cdot ds = \int x^2 y \cdot ds$

We saw in unit 9 on arc length that *ds* can be written as $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt$ so that, now:

M = $\int x^2 y \cdot \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt$

Considering the curve around which we will integrate, we can see that:

x = 2·cosθ
y = 2·sinθ

And if we let t = θ (imagining we walk steadily around the curve as time passes)

We have:

M = $\int 4cos^2\theta \cdot 2sin\theta \cdot \sqrt{(-2sin\theta)^2 + (2cos\theta)^2} \cdot d\theta$ with boundaries of 0 and π.

= $\int_0^{\pi} 16cos^2\theta \cdot sin\theta \cdot d\theta$

= $\frac{-16cos^3\theta}{3}$ with boundaries of 0 and π

= $\frac{32}{3}$

As per usual, extending this into three dimensions is straightforward. Imagine we move through space in little increments *dx*, *dy*, *dz*, as small intervals of time, *dt*, elapse. The length of the path through which we travel is then:

ds = $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$

The field itself through which we travel is some function of spatial coordinates (x,y,z):

F = f(x,y,z)

And, again, the relevant positions are functions of our parameter, *t*, so that:

F = f(x(t),y(t),z(t))

Altogether gives us the line integral

$$\int \mathrm{f(x(t), y(t), z(t))} \cdot \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$$

Or, more succinctly, where $\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$

$$\int f(\boldsymbol{r}(t)) \cdot |\boldsymbol{r}'(t)| \cdot dt$$

Example: Find the line integral for the function $\mathbf{F}(x,y) = xy^2\hat{\boldsymbol{\imath}} - x^2\hat{\boldsymbol{\jmath}}$ along the curve $\mathbf{r}(t) = t^3\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}$ within the boundaries of $0 \leq t \leq 1$.

From the position function, we can see that $x = t^3$ and $y = t^2$ so that

$\mathbf{F}(x,y) = t^7\hat{\boldsymbol{\imath}} - t^6\hat{\boldsymbol{\jmath}}$ and $\mathbf{r}' = 3t^2\hat{\boldsymbol{\imath}} - 2t\hat{\boldsymbol{\jmath}}$

$\int_0^1 (t^7\hat{\boldsymbol{\imath}} - t^6\hat{\boldsymbol{\jmath}}) \cdot (3t^2\hat{\boldsymbol{\imath}} + 2t\boldsymbol{j})$

$= \int_0^1 (3t^9 - 2t^7) \cdot dt$

$= \frac{1}{20}$

In mechanics, an early application is the concept of *work*, where work is the line integral taken through the vector field of force, $\boldsymbol{F}$.

$$W = \int \mathbf{F}(x(t), y(t), z(t)) \cdot \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2 + (\tfrac{dz}{dt})^2} \cdot dt$$

or

$$W = \int \boldsymbol{F}(\boldsymbol{r}(t)) \cdot |\boldsymbol{r}'(t)| \cdot dt$$

or, simply

$$W = \int \boldsymbol{F} \cdot d\boldsymbol{r}$$

Example: The position of an object with mass *m* at time *t* is given by $\mathbf{r}(t) = at^2\hat{\boldsymbol{\imath}} + bt^3\hat{\boldsymbol{\jmath}}$ when $0 \leq t \leq 1$. Find the work input by the net force during this time.

From differentiating the position function, we have $\mathbf{v}(t) = 2at\hat{\boldsymbol{\imath}} + 3bt^2\hat{\boldsymbol{\jmath}}$ and $\mathbf{a}(t) = 2a\hat{\boldsymbol{\imath}} + 6bt\hat{\boldsymbol{\jmath}}$

The net force is then $\mathbf{F}(t) = 2ma\hat{\boldsymbol{\imath}} + 6mbt\hat{\boldsymbol{\jmath}}$.

$W = \int \boldsymbol{F}(\boldsymbol{r}(t)) \cdot |\boldsymbol{r}'(t)| \cdot dt$

$= \int_0^1 (2ma\hat{\boldsymbol{\imath}} + 6mbt\hat{\boldsymbol{\jmath}}) \cdot (2at\hat{\boldsymbol{\imath}} + 3bt^2\hat{\boldsymbol{\jmath}}) \cdot dt$

$= \int_0^1 4ma^2t + 18mb^2t^3 \cdot dt$

$= 2ma^2 + \frac{9}{2}mb^2$

If we take the equation

$$\int f(x(t), y(t), z(t)) \cdot \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$$

between two points along a curve, *a* and *b*, we have:

$$\int_a^b \nabla f \cdot |\boldsymbol{r}'(t)| \cdot dt$$

when expanded becomes:

$$\int_a^b (\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}) \cdot dt$$

which is the chain rule expansion of

$$\int_a^b \frac{d}{dt}(f(\boldsymbol{r}(t)) \cdot dt$$

and leads finally to

$$f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

which is to say

$$\int_a^b \nabla f \cdot d\boldsymbol{r} = f(b) - f(a)$$

Conceptually, this is stating that integrating the gradient of some function along a line from point *a* to point *b* provides the change in that function between the two points.

This agrees with what you may have already seen in electrostatics:

If we are accustomed to using $\vec{E} = -\frac{dV}{dR}$ and the other side of this coin is $\Delta V = -\int \vec{E} \cdot dr$

Then starting with $\vec{E} = -\nabla V$ we can produce $\Delta V = \int \nabla V \cdot dr$

$\int_a^b \nabla f \cdot d\boldsymbol{r} = f(b) - f(a)$

shows that the integral always yields the same result regardless of the path taken. If this is true for the gradient of a function, it is equally true for a conservative vector field from which the gradient can be found. In other words, evaluating a line integral over a conservative vector field always produces the same result, regardless of the path taken.

We also have that

$\int_b^a \nabla f \cdot d\boldsymbol{r} = f(a) - f(b)$

so that if we take some path from *a* to *b* and then some other path from *b* to *a*, we have

$[f(b) - f(a)] + [f(a) - f(b)] = 0.$

In other words, the line integral taken through a conservative vector field around a closed loop always has a value of zero. In physics, the work done by a conservative force around a closed loop is therefore zero.

To determine whether or not a given vector field is conservative, imagine that we do have some conservative field, $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ so that there is some scalar function *f(x,y)* such that $\nabla f = \vec{F}$.

Therefore, $P = \frac{\partial f}{\partial x} = f_x$ and $Q = \frac{\partial f}{\partial y} = f_y$

And $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Clairaut's theorem tells us that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Which means $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

As a result, the vector field function $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ is conservative if it can be shown that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Example: Is the vector field $\vec{F}(x, y) = (3 + 2xy)\hat{\boldsymbol{\imath}} + (x^2 - 3y^2)\hat{\boldsymbol{\jmath}}$ a conservative field?

If $P = (3 + 2xy)$, then $\frac{\partial P}{\partial y} = 2x$

If $Q = (x^2 - 3y^2)$, then $\frac{\partial Q}{\partial x} = 2x$

The two are equal, so the vector field is conservative. How can we determine the *f* which corresponds to this vector field $\vec{F}$, such that $\vec{F} = \nabla f$?

If $\nabla f = \vec{F}$ and $\vec{F}(x, y) = (3 + 2xy)\hat{\boldsymbol{\imath}} + (x^2 - 3y^2)\hat{\boldsymbol{\jmath}}$ then:

$\frac{\partial f}{\partial x} = 3 + 2xy$ and $\frac{\partial f}{\partial y} = (x^2 - 3y^2)$

Integrating the first

$f(x,y) = 3x + x^2y + C_1$ where C_1 must depend upon *y* because it cannot depend upon x

Let us therefore let $C_1 = g(y)$ so that $f(x,y) = 3x + x^2y + g(y)$

Differentiating this last equation produces $\frac{\partial f}{\partial y} = x^2 + g'(y)$

We already know that $\frac{\partial f}{\partial y} = (x^2 - 3y^2)$ from above, so that $g'(y) = -3y^2$ and $g(y) = -y^3 + C_2$

Putting this back in as C_1 yields the solution:

$f(x,y) = 3x + x^2y - y^3 + C_2$

This is easy to confirm by taking the gradient of the function as seeing it results in $\vec{F}$.

You can also take the alternate route by integrating $\frac{\partial f}{\partial y}$ and continuing to reach the same result.

Lastly, by Newton's second law, we have $\mathbf{F} = m\mathbf{a} = m\cdot\mathbf{r}''$

If we had defined work as $W = \int \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \cdot dt$

Then $W = \int m \cdot \mathbf{r}''(t) \cdot \boldsymbol{r}'(t) \cdot dt$

By the product rule $\frac{d}{dt}(\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)) = 2\cdot \mathbf{r}''(t) \cdot \boldsymbol{r}'(t)$

Substitution then provides $\frac{m}{2}\int_a^b \frac{d}{dt}\left(\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(t)\right) \cdot dt = \frac{m}{2}\int_a^b \frac{d}{dt}\left(\boldsymbol{r}'^{\mathbf{2}}(t)\right) \cdot dt$

Yielding $W = \frac{m}{2}[\boldsymbol{r}'^{\mathbf{2}}(b) - \boldsymbol{r}'^{\mathbf{2}}(a)] = \frac{1}{2} m\mathbf{v}(b)^2 - \frac{1}{2} m\mathbf{v}(a)^2$

Which is, of course, stating that the work input equals the change in kinetic energy of the object into which the work is input.

If we define a scalar function called potential energy as U = f(x,y,z) such that $\mathbf{F} = -\nabla U$ again recalling the restriction that this is true for conservative forces:

Then $W = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{r}$

$= -\int_a^b \nabla U \cdot d\boldsymbol{r}$

= U(a) – U(b)

If also W = K(b) – K(a),

then U(a) + K(a) = U(b) + K(b)

Showing mechanical energy is conserved when work in input by conservative forces only.