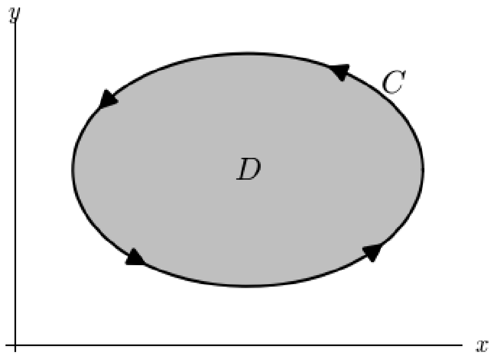
Green's Theorem



As shown in the diagram above, Green's theorem is a relationship between the line integral around a closed path (C) and the double-integral enclosed by that path (D) such that:

$$\int (P \cdot dx + Q \cdot dy) = \iint (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cdot dA$$

The proof of this in not particularly instructive, so I will leave it for the end as an appendix.

Example 1: Find the line integral for the function $\vec{F} = y^2\boldsymbol{\hat{\imath}} + x^2y\boldsymbol{\hat{\jmath}}$ around the square with vertices of (0,0), (5,0), (5,4), and (0,4).

Starting at the origin and traveling counter-clockwise, the line integrals would be:

$\int_0^5 y^2 \cdot dx = 0^2 \cdot (5 - 0) = 0$

$\int_0^4 x^2 y \cdot dy = 5^2 \cdot (\frac{4^2}{2} - \frac{0^2}{2}) = 200$

$\int_5^0 y^2 \cdot dx = 4^2 \cdot (0 - 5) = -80$

$\int_4^0 x^2 y \cdot dy = 0^2 \cdot (\frac{0^2}{2} - \frac{4^2}{2}) = 0$ for a total of 120

We should get the same result using Green's theorem. Let $P = y^2$ and $Q = x^2y$ so that

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2xy - 2y.$

$\int_0^4 \int_0^5 (2xy - 2y) dx \cdot dy = \int_0^4 15y \cdot dy = 120$ as before

Example 2: Find the line integral around the circle $x^2 + y^2 = 4$ in the field $\mathbf{F} = y^3\hat{\boldsymbol{\imath}} - x^3\hat{\boldsymbol{\jmath}}$.

Letting $P = y^3$ and $Q = -x^3$ we have $\frac{\partial P}{\partial y} = 3y^2$ and $\frac{\partial Q}{\partial x} = -3x^2$

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot dA = \iint (-3x^2 - 3y^2) \cdot dA = -3\int_0^{2\pi} \int_0^2 R^2cos^2\theta + R^2sin^2\theta \cdot R \cdot dR \cdot d\theta$$

$$= -12\int_0^{2\pi} (1)d\theta = -24\pi$$

Green's theorem can also be used "in reverse" in situations when a line integral is used to determine the area.

If area = $\iint dA$ then, by Green's theorem: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

For instance, if $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 0$, then $Q(x,y) = x$ and $P(x,y) = 0$

making the area equation (from the left side of Green's theorem)

$A = \oint x \cdot dy$

Other common versions are: $A = \oint -y \cdot dx$ and $A = \frac{1}{2}\oint x \cdot dy - y \cdot dx$

Example 3: An ellipse with semimajor and semiminor axes of *a* and *b* has the parametric equations:

$x = a\cdot\cos(t)$ and $y = b\cdot\sin(t)$

Find the area of the ellipse.

Using $A = \frac{1}{2}\oint x \cdot dy - y \cdot dx$ we have

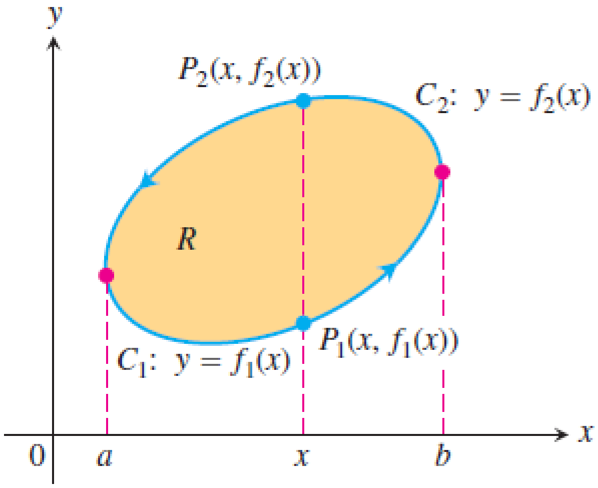
$dx = -a\cdot\sin(t)\cdot dt$ and $dy = b\cdot\cos(t)\cdot dt$

$$A = \frac{1}{2}\oint x \cdot dy - y \cdot dx = \frac{1}{2}\oint \mathrm{a} \cdot \cos(\mathrm{t}) \cdot \mathrm{b} \cdot \cos(\mathrm{t}) \cdot \mathrm{dt} - \mathrm{b} \cdot \sin(\mathrm{t}) \cdot -\mathrm{a} \cdot \sin(\mathrm{t}) \cdot \mathrm{dt}$$

$$= \frac{1}{2}\oint_0^{2\pi} ab \cdot dt = \pi ab$$

We saw in unit eleven that, for conservative fields, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Accordingly, the right side of Green's theorem is zero. If we consider the left side of the equation as the line integral of work, $\oint \vec{F} \cdot d\vec{r}$, we see again that the work done by a conservative force around a closed loop is zero.

Appendix:



Going from a to b, we have:

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy = P(x, f_2(x)) - P(x, f_1(x))$$

Integrating this with respect to *x* from *a* to *b* (to complete the double integral) is:

$\int_a^b P\left(x, f_2(x)\right) - P\left(x, f_1(x)\right) \cdot dx =$

$-\int_b^a P\left(x, f_2(x)\right) \cdot dx - \int_a^b P\left(x, f_1(x)\right) \cdot dx =$

$-\int_{C_2} P \cdot dx - \int_{C_1} P \cdot dx =$

$-\oint_C P \cdot dx$ so that $\oint_C P \cdot dx = \iint -\frac{\partial P}{\partial y} dA$

The process is essentially the same to show $\oint_C Q \cdot dy = \iint \frac{\partial Q}{\partial x} dA$

The negative sign is absent because the region is now of type II and the boundaries *c* and *d* are symmetric. The sum of the two results is Green's theorem.