Curl and Divergence

Curl and divergence, like gradient, are known in vector calculus as *operators*. They take some field as an input and return some other field as an output. For instance, we've seen that the gradient operator can take the scalar field:

$$f(x,y,z) = 12xy + 6yz^2 - 8xz^3$$

and produce

$$\nabla f = (12y - 8z^3)\hat{\boldsymbol{\imath}} + (12x + 6z^2)\,\hat{\boldsymbol{\jmath}} + (12yz - 24xz^2)\,\hat{\boldsymbol{k}}$$

The easiest way to remember the curl operator is with the notation:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

as if the curl of the vector field, **F**, is the cross product of a gradient and the vector field. It's not really a cross-product, but it can be generated in the same way a cross product can be generated, by finding the determinant of a matrix. If $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}$, then:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{matrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{matrix}$$

$$= \frac{\partial R}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial P}{\partial z}\hat{\boldsymbol{\jmath}} + \frac{\partial Q}{\partial x}\hat{\boldsymbol{k}} - \frac{\partial P}{\partial y}\hat{\boldsymbol{k}} - \frac{\partial Q}{\partial z}\hat{\boldsymbol{\imath}} - \frac{\partial R}{\partial x}\hat{\boldsymbol{\jmath}}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\boldsymbol{k}}$$

Example: Find the curl of $\mathbf{F}(x,y,z) = xz\hat{\boldsymbol{\imath}} + xyz\hat{\boldsymbol{\jmath}} - y^2\hat{\boldsymbol{k}}$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{matrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{matrix}$$

$$= \left(\frac{\partial -y^2}{\partial y} - \frac{\partial xyz}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial xz}{\partial z} - \frac{\partial -y^2}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial xyz}{\partial x} - \frac{\partial xz}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$= (-2y - xy)\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + yz\,\hat{\boldsymbol{k}}$$

If we take the curl of a gradient, we have:

$$\text{curl}(\nabla f) = \quad \nabla \times (\nabla f) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{\boldsymbol{k}}$$

$$= 0 \text{ by Clairaut's theorem that } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Conservative fields are such that $\mathbf{F} = \nabla f$, so the above shows that the curl in a conservative field is always zero.

Example: Show that $\mathbf{F}(x,y,z) = (xy^2z^2)\,\hat{\boldsymbol{\imath}} + (x^2yz^2)\,\hat{\boldsymbol{\jmath}} + (x^2y^2z)\,\hat{\boldsymbol{k}}$ is a conservative field.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= (2x^2yz - 2x^2yz)\hat{\boldsymbol{\imath}} + (2xy^2z - 2xy^2z)\,\hat{\boldsymbol{\jmath}} + (2xyz^2 - 2xyz^2)\,\hat{\boldsymbol{k}}$$

$$= 0$$

The curl operator is useful in physical situations that rotate, for example a solid disk spinning with an angular velocity $\boldsymbol{\omega}$ has points along the disk which move with tangential velocity $\mathbf{v}$ such that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R}$.

For a disk rotating counter-clockwise in the plane of the page, this gives us:
$\mathbf{v} = -\omega y\,\boldsymbol{i} + \omega x\,\hat{\boldsymbol{\jmath}}$

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \frac{\partial 0}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial - \omega y}{\partial z}\hat{\boldsymbol{\jmath}} + \frac{\partial \omega x}{\partial x}\hat{\boldsymbol{k}} - \frac{\partial - \omega y}{\partial y}\hat{\boldsymbol{k}} - \frac{\partial \omega x}{\partial z}\hat{\boldsymbol{\imath}} - \frac{\partial 0}{\partial x}\hat{\boldsymbol{\jmath}}$$

$$= 2\omega\,\hat{\boldsymbol{k}}$$

Conversely, systems which are irrotational (like $\mathbf{v} = 5\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 6\,\hat{\boldsymbol{k}}$) have a curl of zero and field is said to be *irrotational*.

In the same way that the curl operator can be remembered as something like a cross product, the *divergence* operator can be remembered as something like the dot product:

$$\text{div } \mathbf{F} = \nabla \bullet \mathbf{F}$$

Recalling that $\nabla = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})$ for the vector field $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}$ therefore

$\text{div } \mathbf{F} = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})$

Example: What is the divergence of the vector field $\mathbf{F}(x,y,z) = xz\,\hat{\boldsymbol{\imath}} + xyz\,\hat{\boldsymbol{\jmath}} - y^2\,\hat{\boldsymbol{k}}$?

$\text{div } \mathbf{F} = (\frac{xz}{\partial x} + \frac{xyz}{\partial y} + \frac{y^2}{\partial z}) = z + xz$

In the same way that the dot product produces a scalar result, the divergence operator produces a scalar field.

If we take the divergence of curl, we have:

$\text{div (curl } \mathbf{F}) = \nabla \bullet (\nabla \times \mathbf{F})$

$= \frac{\partial}{\partial x}(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$

$= 0$ by Clairaut's theorem that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

We can also take the divergence of a gradient of a scalar field with:

$\text{div } (\nabla f) = \nabla \bullet (\nabla f)$

$= \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) + \frac{\partial}{\partial z}(\frac{\partial f}{\partial z})$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Often written as $\nabla^2 f$, this is called the *Laplace operator* or *Laplacian*.

Lastly, we can write Green's theorem in terms of these new operators:

If we have $\mathbf{F} = P\,\hat{\boldsymbol{\imath}} + Q\,\hat{\boldsymbol{\jmath}}$

Then curl $\mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\boldsymbol{k}}$

curl $\mathbf{F}\cdot\hat{\boldsymbol{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

Which then goes into Green's theorem as:

$$\int (\boldsymbol{F}\cdot d\boldsymbol{r}) = \iint (\text{curl}\,\mathbf{F}\cdot\hat{\boldsymbol{k}})\cdot dA$$

If **F** is some vector field in the plane of the page, curl $\mathbf{F}\cdot\hat{\boldsymbol{k}}$ represents the component of the curl into or out of the page and $\iint(\text{curl}\,\mathbf{F}\cdot\hat{\boldsymbol{k}})\cdot dA$ integrates these components over the area enclosed by the line integral on the left side of the equation.

We can also write Green's theorem in terms of the vector *normal* to the path in Green's theorem. As seen in unit 8, the unit tangent vector for the equation: $\mathbf{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$ is:

$\mathbf{T}(t) = \frac{x'(t)}{|\boldsymbol{r}'(t)|}\hat{\boldsymbol{\imath}} + \frac{y'(t)}{|\boldsymbol{r}'(t)|}\hat{\boldsymbol{\jmath}}$ making the unit normal vector perpendicular to this:

$\mathbf{n}(t) = \frac{y'(t)}{|\boldsymbol{r}'(t)|}\hat{\boldsymbol{\imath}} - \frac{x'(t)}{|\boldsymbol{r}'(t)|}\hat{\boldsymbol{\jmath}}$

Let $\mathbf{G} = Q\,\hat{\boldsymbol{\imath}} - P\,\hat{\boldsymbol{\jmath}}$

Then $\text{div}(\mathbf{G}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

and $\mathbf{G}\cdot\mathbf{n} = Q\cdot\frac{y'(t)}{|\boldsymbol{r}'(t)|} + P\cdot\frac{x'(t)}{|\boldsymbol{r}'(t)|}$

Then taking the path integral, we have

$\oint \mathbf{G}\cdot\mathbf{n}\cdot ds = \int Q\cdot y'(t)\cdot dt + P\cdot x'(t)\cdot dt = \int Q\cdot dy + P\cdot dx$

which is also $\iint(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\cdot dA$ by Green's theorem

Therefore $\oint \mathbf{G}\cdot\mathbf{n}\cdot ds = \iint \text{div}(\mathbf{G})\cdot dA$