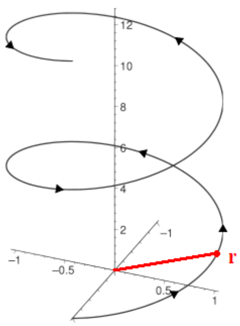
Parametric Surfaces

In unit 9, we saw it is often easiest to define a curve in space with parametric equations, such as:

$\mathbf{r}(t) = \cos(t)\cdot\hat{\boldsymbol{\imath}} + \sin(t)\cdot\hat{\boldsymbol{\jmath}} + t\cdot\hat{\boldsymbol{k}}$ representing all the points of a helix spiraling along the z-axis

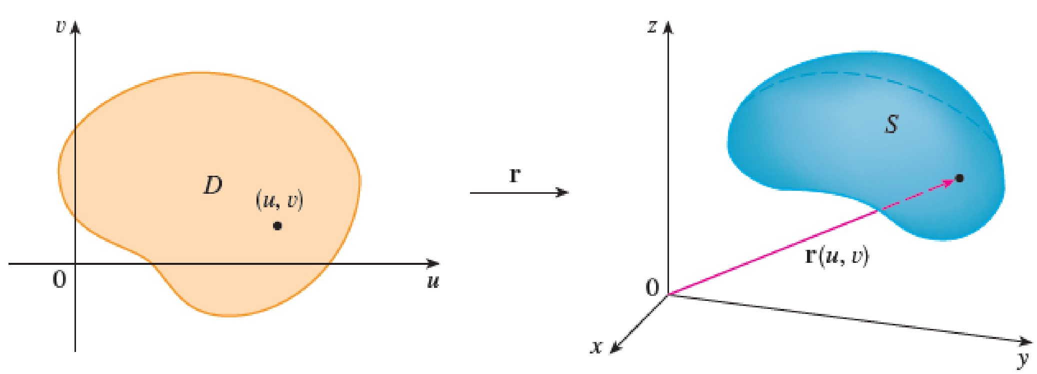
What the equation above indicates is that where you are in space (the **r** position vector) depends upon some functions, namely cos(t), sin (t) and t, which in turn depends upon some value, *t*. We can get away with using only one parameter, *t*, because the helix is a one-dimensional curve.



But how can we represent a two-dimensional surface parametrically? It would make sense that if a one-dimensional curve requires one parameter, a two-dimensional surface would need two (conventionally *u* and *v*) used in the position function such that:

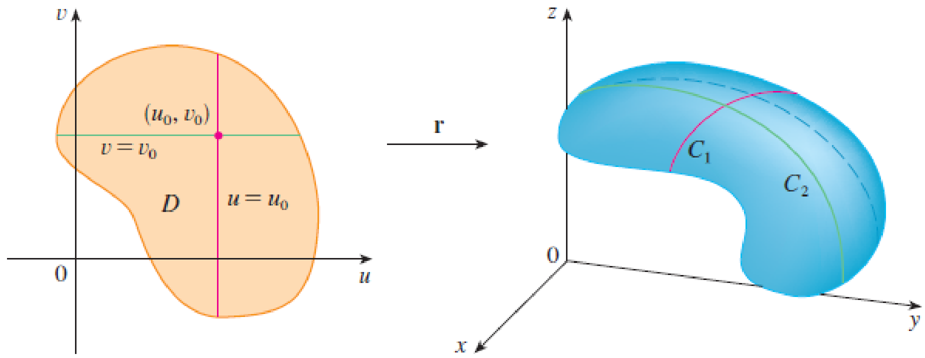
$\mathbf{r}(u,v) = f(u,v)\cdot\hat{\boldsymbol{\imath}} + g(u,v)\cdot\hat{\boldsymbol{\jmath}} + h(u,v)\cdot\hat{\boldsymbol{k}}$

What the equation above means is that we can define a series of points in space (the function **r**) that depend upon the f, g, and h functions and the values of those functions depend upon where you are in the *uv plane* (shown below). The tricky part is realizing that the *uv plane* is not some spatial entity like the xy-plane; it's a kind of abstract function space.



You can imagine walking around the *uv plane* and every time you stand at a given uv-coordinate, you look down and see three coordinates (x, y, and z) for actual 3D space. Imagine putting a tiny hovering drone at those coordinates, then moving to a slightly different spot on the uv plane, reading the x, y, and z values below you, placing another drone there, etc. Once you have walked over the entire uv plane and done this, the drone fleet will map-out the curve, *S*.

Imagine clearing the skies and now, instead, walking entirely along the *v* direction of this abstract *uv plane*. You would place a series of drones that would only form a curve along the full surface, *S*. This is called a *grid curve*. Likewise, you could walk along the *u* direction of the *uv plane* and map-out a different grid curve.



Example: A cone is a surface which follows the equation: z = $\sqrt{x^2 + y^2}$ for $0 \leq z \leq 5$. Write this same surface with a parametric equation.

The straightforward answer would be:

$\mathbf{r}(x,y) = x\boldsymbol{\cdot}\hat{\boldsymbol{\imath}} + y\boldsymbol{\cdot}\hat{\boldsymbol{\jmath}} + \sqrt{x^2 + y^2}\boldsymbol{\cdot}\hat{\boldsymbol{k}}$

Another possibility is:

If we let z = u, then $u^2 = x^2 + y^2 = u^2\cdot\cos^2\theta + u^2\cdot\sin^2\theta$

allowing $x = u\cdot\cos\theta$ and $y = u\cdot\sin\theta$

Resulting in $\mathbf{r}(u,\theta) = u\cdot\cos\theta\boldsymbol{\cdot}\hat{\boldsymbol{\imath}} + u\cdot\sin\theta\boldsymbol{\cdot}\hat{\boldsymbol{\jmath}} + u\boldsymbol{\cdot}\hat{\boldsymbol{k}}$ with the two parameters u and θ

Here, an obvious grid curve would be one where u is held constant and θ varies from 0 to 2π, creating the circle at some altitude, u.

Example: Write the parametric equation for a sphere with a radius of 5.

We know that $x^2 + y^2 + z^2 = 5$. As seen in unit 19, to convert from Cartesian to spherical coordinates, we imagine a point on the sphere at an angle φ above the xy-plane casting a shadow on the xy-plane with length 5·cosφ. Rotating this shadow from 0 to 2π in the xy-plane gives us:

$x = 5\cdot\cos\theta\cdot\cos\phi$ $y = 5\cdot\sin\theta\cdot\cos\phi$ $z = 5\cdot\sin\phi$

$\mathbf{r}(\varphi, \theta) = 5\cdot\cos\theta\cdot\cos\phi\boldsymbol{\cdot}\hat{\boldsymbol{\imath}} + 5\cdot\sin\theta\cdot\cos\phi\boldsymbol{\cdot}\hat{\boldsymbol{\jmath}} + 5\cdot\sin\phi\boldsymbol{\cdot}\hat{\boldsymbol{k}}$

Likewise, converting from Cartesian to cylindrical coordinates gives us a way of writing surfaces of revolution with parametric equations. For instance, take the simple function,

$y = 5$ for $0 \leq x \leq 10$.

Rotating this around the yz-plane gives us, from 0 to 2π

$y = 5 \cdot \cos\theta$ and $z = 5 \cdot \sin\theta$ and $x = x$

So the parametric equation would be

$\mathbf{r}(x,\theta) = x \cdot \hat{\boldsymbol{\imath}} + 5 \cdot \cos\theta \cdot \hat{\boldsymbol{\jmath}} + 5\sin\theta \cdot \hat{\boldsymbol{k}}$

Which works just as well for something like $y = 5\sqrt{x}$:

$\mathbf{r}(x,\theta) = x \cdot \hat{\boldsymbol{\imath}} + 5\sqrt{x} \cdot \cos\theta \cdot \hat{\boldsymbol{\jmath}} + 5\sqrt{x} \cdot \sin\theta \cdot \hat{\boldsymbol{k}}$

Recall that walking through the uv-plane at a constant value of u and a changing v maps out a gird curve in space along the surface, *S*. We can call this **r** as a function of v, so that, at some point on the surface, (u_0, v_0):

$\mathbf{r}(u_0, v_0) = x \cdot \hat{\boldsymbol{\imath}} + y \cdot \hat{\boldsymbol{\jmath}} + z \cdot \hat{\boldsymbol{k}}$ and

$$\mathbf{r}_v = \frac{d\boldsymbol{r}}{dv} = \frac{\partial x}{\partial v} \cdot \hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial v} \cdot \hat{\boldsymbol{\jmath}} + \frac{\partial z}{\partial v} \cdot \boldsymbol{k}$$

which represents the slope of the line tangent to the curve at (u_0, v_0) along the v direction

Likewise,

$$\mathbf{r}_u = \frac{d\boldsymbol{r}}{du} = \frac{\partial x}{\partial u} \cdot \hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial u} \cdot \hat{\boldsymbol{\jmath}} + \frac{\partial z}{\partial u} \cdot \boldsymbol{k}$$

$\mathbf{r}_v$ and $\mathbf{r}_u$ then form a plane which can be defined by the vector normal to that plane, $\mathbf{r}_v \times \mathbf{r}_u$.

Example: For the parametric equations

$x = u + v$ $y = 3u^2$ $z = u - v$

Find an equation of the tangent plane at the point (2, 3, 0).

First, solve the equations $2 = u + v$ $3 = 3u^2$ and $0 = u - v$

to find u =1 and v = 1

We have $\mathbf{r} = (u + v)\cdot\hat{\boldsymbol{\imath}} + (3u^2)\cdot\hat{\boldsymbol{\jmath}} + (u - v)\cdot\widehat{\boldsymbol{k}}$

$\mathbf{r_v} = \frac{d\boldsymbol{r}}{dv} = \frac{\partial x}{\partial v}\cdot\hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial v}\cdot\hat{\boldsymbol{\jmath}} + \frac{\partial z}{\partial v}\cdot\boldsymbol{k} = 1\cdot\hat{\boldsymbol{\imath}} + 0\cdot\hat{\boldsymbol{\jmath}} - 1\cdot\widehat{\boldsymbol{k}}$

$\mathbf{r_u} = \frac{d\boldsymbol{r}}{du} = 1\cdot\hat{\boldsymbol{\imath}} + 6u\cdot\hat{\boldsymbol{\jmath}} + 1\cdot\boldsymbol{k} = 1\cdot\hat{\boldsymbol{\imath}} + 6\cdot\hat{\boldsymbol{\jmath}} + 1\cdot\boldsymbol{k}$

$$\mathbf{r_v \times r_u} = \begin{matrix} \hat{\boldsymbol{\imath}} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 1 & 6 & 1 \end{matrix} = 6\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 6\boldsymbol{k}$$

Referring to unit 6, we see the way to convert this into the equation of a plane is with:

$6(x - x_0) - 2(y - y_0) + 6(z - z_0) = 0$

$6(x - 2) - 2(y - 3) + 6(z - 0) = 0$

$3x - y + 3z - 3 = 0$

Imagine taking this equation for the tangent plane and shrinking down the area to $\Delta u\cdot\Delta v$. This would give us an approximate area for the corresponding patch on the curve, *S*. We could then find the sum of these patches over the whole uv-plane to find the surface area of *S*.

Shrinking $\Delta u\cdot\Delta v$ to $du\cdot dv$ gives us the actual equation for the area of the surface:

Area of S = $\iint |\boldsymbol{r_u} \; \boldsymbol{x} \; \boldsymbol{r_v}| \cdot du \cdot dv$

However, a more useful form is in terms of x, y, and z rather than in terms of u and v. Suppose we have some shape above the xy-plane such that z = f(x,y). Given that x = x and y = y,

$\mathbf{r_x} = \frac{df}{dx} = \frac{d}{dx}\,(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z(x,y)\boldsymbol{k}) = \hat{\boldsymbol{\imath}} + \frac{\partial z}{dx}\widehat{\boldsymbol{k}}$

Likewise, $\mathbf{r_y} = \hat{\boldsymbol{\jmath}} + \frac{\partial z}{dx}\widehat{\boldsymbol{k}}$

The cross product $\mathbf{r_x \times r_y} = -\frac{\partial z}{dx}\hat{\boldsymbol{\imath}} - \frac{\partial z}{dy}\hat{\boldsymbol{\jmath}} + 1\widehat{\boldsymbol{k}}$

The magnitude of a three dimension vector is just $\sqrt{x^2 + y^2 + z^2}$

so the magnitude of this cross product is $\sqrt{1^2 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$

and, using the same form as the area equation above:

Area of S = $\iint \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot dx \cdot dy$

Example: Find the area of the plane 3x + 2y + z = 6 that lies in the first octant.

z = 6 – 3x – 2y

$\frac{\partial z}{\partial x}$ = -3

$\frac{\partial z}{\partial y}$ = -2

Area = $\int_0^3 \int_0^{2-\frac{2y}{3}} \sqrt{1 + (-3)^2 + (-2)^2} \cdot dx \cdot dy$

= $\int_0^3 \int_0^{2-\frac{2y}{3}} \sqrt{14} \cdot dx \cdot dy$

= $3\sqrt{14}$