Surface Integrals

Earlier, we saw the equation for arc length could be written as

$$\int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$$

and, further, that we could integrate this along some scalar or vector field to produce a line integral:

$$\int \mathrm{f(x(t), y(t), z(t))} \cdot \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$$

In unit 24, we derived equations for surface area:

Area = $\iint |\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}| \cdot du \cdot dv$ and Area = $\iint \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot dx \cdot dy$

If we, likewise, integrate these across some scalar field, we would have a *surface integral*:

$\iint f(\boldsymbol{r}(u,v)) \cdot |\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}| \cdot du \cdot dv$ and $\iint f(x,y,g(x,y)) \cdot \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot dx \cdot dy$

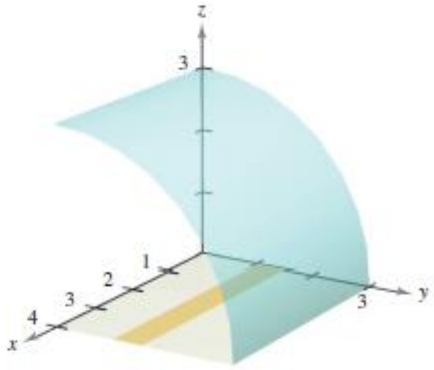
Of course, the second of these is specific to some z-axis function depending upon x and y coordinates, but there is nothing special about the z-axis, so it could just as well be:

$$\iint f(g(y,z),y,z)\sqrt{1 + (\frac{\partial x}{\partial y})^2 + (\frac{\partial x}{\partial z})^2} \cdot dy \cdot dz$$

or

$$\iint f(x,g(x,z),z)\sqrt{1 + (\frac{\partial y}{\partial x})^2 + (\frac{\partial y}{\partial z})^2} \cdot dx \cdot dz$$

Example: Evaluate the surface integral $\iint (x+z)dS$ where S is a quarter-cylinder along the x-axis between x = 0 and x = 4 such that $y^2 + z^2 = 9$.



Let's first use $\iint f(x,y,g(x,y)) \cdot \sqrt{1+ (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot dx \cdot dy$

If $z = \sqrt{9 - y^2}$ then $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = -y(9 - y^2)^{-3/2}$

$\iint (x + \sqrt{9-y^2}) \cdot \sqrt{1+ y^2(9-y^2)^{-1}} \cdot dx \cdot dy$

$= \iint (x + \sqrt{9-y^2}) \cdot \frac{3}{\sqrt{9-y^2)}} \cdot dx \cdot dy$

$= 3 \cdot \iint (\frac{x}{\sqrt{9-y^2}} + 1) \cdot dx \cdot dy$

$= 3 \cdot \int_0^3 \int_0^4 (\frac{x}{\sqrt{9-y^2}} + 1) \cdot dx \cdot dy$

$= 3 \cdot \int_0^3 (\frac{x^2}{2\sqrt{9-y^2}} + x) \cdot dy$ from x = 0 to x = 4

$= 12 \cdot \int_0^3 (\frac{2}{\sqrt{9-y^2}} + 1) \cdot dy$

$= 12 \cdot (2 \cdot sin^{-1}(\frac{y}{3}) + y)$ from y = 0 to y = 3

$= 36 + 24(\frac{\pi}{2})$

$= 36 + 12\pi$

We should get the same result using $\iint f(\boldsymbol{r}(u,v)) \cdot |\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}| \cdot du \cdot dv$

For the quarter-cylinder, we can write the surface as:

$\mathbf{r}(x, \theta) = x\cdot\hat{\boldsymbol{\imath}} + 3\cos\theta\cdot\hat{\boldsymbol{\jmath}} + 3\sin\theta\cdot\hat{\boldsymbol{k}}$

$\mathbf{r_x} = \hat{\boldsymbol{\imath}}$ and $\mathbf{r_\theta} = -3\sin\theta\cdot\hat{\boldsymbol{\jmath}} + 3\cos\theta\cdot\hat{\boldsymbol{k}}$

$\boldsymbol{r_x}\ \boldsymbol{x}\ \boldsymbol{r_\theta} = -3\cos\theta\cdot\hat{\boldsymbol{\jmath}} - 3\sin\theta\cdot\hat{\boldsymbol{k}}$

so the magnitude of the cross product, $|\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}| = 3$

$\int_0^4 \int_0^{\pi/2} (x + 3sin\theta) \cdot 3 \cdot d\theta \cdot dx =$

$\int_0^4 (3x\theta - 9cos\theta) \cdot dx$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$

$\int_0^4 (3x\frac{\pi}{2} + 9) \cdot dx =$

$12\pi + 36$ as before

The surface integral over a vector field is known as the *flux integral*, or more commonly, just the *flux* through the surface. For instance, if we had a sieve in which water was draining-out, we could relate the flux of water through the sieve to the rate at which the volume of water inside was decreasing. But the relevant fluid flow would be normal to the surface of the sieve, because flow tangential to the surface is not changing how much volumetric water the sieve is holding.

Therefore, we take the surface integral of the dot product between the field and the normal unit vector, **n**:

$\iint \boldsymbol{F} \bullet \boldsymbol{n} \cdot dS$ as the flux across the surface, S

We have already seen to normal unit vector as:

$\mathbf{n} = \frac{\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}}{|\ r_u\ x\ r_v\ |}$

and, earlier in these notes:

$dS = |r_u\ x\ r_v| \cdot du \cdot dv$

Therefore

$\iint \boldsymbol{F} \bullet \boldsymbol{n} \cdot dS = \iint \boldsymbol{F} \bullet \frac{\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}}{|\ r_u\ x\ r_v\ |} \cdot |r_u\ x\ r_v| \cdot du \cdot dv$

reduces to

Flux = $\iint \boldsymbol{F} \bullet (\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}) \cdot du \cdot dv$

Example: Find the flux of $\mathbf{F} = yz \cdot \hat{\boldsymbol{\imath}} + x \cdot \hat{\boldsymbol{\jmath}} - z^2 \cdot \hat{\boldsymbol{k}}$ through the parabolic cylinder of $y = x^2$ where $0 \le x \le 1$ and $0 \le z \le 4$.

The surface is easy to parameterize with $x = x$, $y = x^2$, and $z = z$, so that:

$\mathbf{r}(x,z) = x \cdot \hat{\boldsymbol{\imath}} + x^2 \cdot \hat{\boldsymbol{\jmath}} + z \cdot \hat{\boldsymbol{k}}$

$\mathbf{r_x} = \hat{\boldsymbol{\imath}} + 2x \cdot \hat{\boldsymbol{\jmath}}$

$\mathbf{r_z} = \hat{\boldsymbol{k}}$

$\boldsymbol{r_x}\ \boldsymbol{x}\ \boldsymbol{r_z} = 2x \cdot \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$

$\boldsymbol{F} \bullet (\boldsymbol{r_u}\ \boldsymbol{x}\ \boldsymbol{r_v}) = 2xzy - x = 2x^3z - x$

$\int_0^4 \int_0^1 (2x^3z - x) \cdot dx \cdot dz = \int_0^4 (\frac{z}{2} - \frac{1}{2}) dz = 2$

We also saw, in unit 24, that $\mathbf{r_x} \text{ x } \mathbf{r_y} = -\frac{\partial z}{dx}\hat{\boldsymbol{\imath}} - \frac{\partial z}{dy}\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$

So for $\mathbf{F}(x,y,z) = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\boldsymbol{k}$

Flux = $\iint (P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\boldsymbol{k}) \bullet \left(-\frac{\partial z}{dx}\hat{\boldsymbol{\imath}} - \frac{\partial z}{dy}\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}\right) \cdot dx \cdot dy$

Flux = $\iint \left(-P\frac{\partial z}{dx} - Q\frac{\partial z}{dy} + R\right) \cdot dx \cdot dy$

Example: Find the flux of the field

$\mathbf{F}(x,y,z) = y\cdot\hat{\boldsymbol{\imath}} + x\cdot\hat{\boldsymbol{\jmath}} + z\cdot\hat{\boldsymbol{k}}$

through the paraboloid $z = 1 - x^2 - y^2$.

$-P\frac{\partial z}{dx} = 2xy$

$-Q\frac{\partial z}{dy} = 2xy$

$R = z = 1 - x^2 - y^2$

Flux = $\iint (4xy + 1 - x^2 - y^2) \cdot dx \cdot dy$ converted to polar coordinates is

$\int_0^{2\pi}\int_0^1 (1 + 4r^2 sin\theta cos\theta - r^2) \cdot r \cdot dr \cdot d\theta =$

$\int_0^{2\pi} (\frac{1}{4} + sin\theta cos\theta) \cdot d\theta = \frac{\theta}{4} + \frac{1}{2}\sin^2\theta$ with bounds of 0 and 2π =

$\frac{\pi}{2}$