Stokes' Theorem

In unit 22, we had Green's theorem as:

$$\int(P \cdot dx + Q \cdot dy) = \iint(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cdot dA$$

a relationship between a path integral (left side of the equation) and a double-integral of the area enclosed by the path (right side).

Stokes' theorem is also a relationship between a line integral and a double-integral of the enclosed area, and is actually a more general version of Green's theorem.

Conceptually, Stokes' theorem states that:

The line integral around the boundary of surface *S* of the tangential component of the field **F** through *S* is equal to the surface integral of *S* of the normal component of the curl of **F**.

So the line integral of a closed path through a field *tangent* to the path would be:

$\oint \boldsymbol{F} \cdot \boldsymbol{T} \cdot ds$ or $\oint \boldsymbol{F} \cdot d\boldsymbol{r}$ d**r** being $\mathbf{r}_2 - \mathbf{r}_1$ which is always tangent to the path

equals

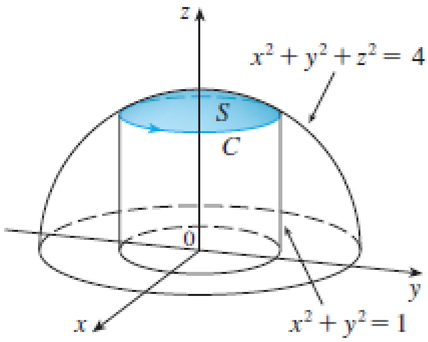
$\iint curl\, \boldsymbol{F} \cdot \boldsymbol{n} \cdot dS$ or $\iint curl\, \boldsymbol{F} \cdot d\boldsymbol{S}$ d**S** being perpendicular to the surface by definition

Stokes' theorem is then:

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \iint curl\, \boldsymbol{F} \cdot d\boldsymbol{S}$$

The proof of this I will put as an appendix at the end.

Example: Use Stokes' theorem to find $\iint curl\, \boldsymbol{F} \cdot d\boldsymbol{S}$ for the field $\mathbf{F}(x,y,z) = xz\cdot\hat{\boldsymbol{\imath}} + yz\cdot\hat{\boldsymbol{\jmath}} + xy\cdot\widehat{\boldsymbol{k}}$ where S is part of the $x^2 + y^2 + z^2 = 4$ hemisphere that lies within the cylinder $x^2 + y^2 = 1$.



The two shapes meet at the intersection of $z^2 = 3$, so at $z = \sqrt{3}$. The vector equation for the curve is therefore:

$\mathbf{r}(\theta) = \cos\theta\cdot\hat{\boldsymbol{\imath}} + \sin\theta\cdot\hat{\boldsymbol{\jmath}} + \sqrt{3}\cdot\widehat{\boldsymbol{k}}$

and

$\mathbf{r}'(\theta) = -\sin\theta\cdot\hat{\boldsymbol{\imath}} + \cos\theta\cdot\hat{\boldsymbol{\jmath}}$

From $\mathbf{F}(x,y,z) = xz\cdot\hat{\boldsymbol{\imath}} + yz\cdot\hat{\boldsymbol{\jmath}} + xy\cdot\widehat{\boldsymbol{k}}$ and substitution, we have

$\mathbf{F}(\mathbf{r}(\theta)) = \sqrt{3}\cos\theta\cdot\hat{\boldsymbol{\imath}} + \sqrt{3}\sin\theta\cdot\hat{\boldsymbol{\jmath}} + \cos\theta\sin\theta\cdot\widehat{\boldsymbol{k}}$

The right side of Stokes' theorem is $\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \oint \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta) \cdot d\theta$

$= \int_0^{2\pi}[(\sqrt{3}\cos\theta\cdot\hat{\boldsymbol{\imath}} + \sqrt{3}\sin\theta\cdot\hat{\boldsymbol{\jmath}} + \cos\theta\sin\theta\cdot\widehat{\boldsymbol{k}})\cdot(-\sin\theta\cdot\hat{\boldsymbol{\imath}} + \cos\theta\cdot\hat{\boldsymbol{\jmath}})] \cdot d\theta$

$= \int_0^{2\pi}(-\sqrt{3}\cos\theta \cdot sin\theta \;+\; \sqrt{3}\cos\theta \cdot sin\theta) \;\cdot d\theta$

$= 0$

In fluid mechanics, let us define the term *flow* as the path integral of fluid velocity tangent to the path:

Flow = $\int_a^b \boldsymbol{v} \cdot d\boldsymbol{r}$

and if the path form a closed loop, we have the *circulation* of the fluid:

circulation = $\oint \boldsymbol{v} \cdot d\boldsymbol{r}$

Using Stokes' theorem, we have

circulation = $\oint \boldsymbol{v} \cdot d\boldsymbol{r} = \iint curl\, \boldsymbol{v} \cdot \boldsymbol{n} \cdot dS$

Showing that the curl of the fluid velocity is a measure of that fluid's rotation around some point with normal vector **n** and is maximum when parallel to **n.**

Stokes' theorem also shows us that if $\oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ in a conservative field, then

$\iint curl\, \boldsymbol{F} \cdot d\boldsymbol{S} = 0$

and curl **F** must also be zero inside a conservative field.

Appendix – specialized proof of Stokes' theorem:

For the field, **F** = P·$\hat{\boldsymbol{\imath}}$ + Q·$\hat{\boldsymbol{\jmath}}$ + R·$\hat{\boldsymbol{k}}$

we take the equation from unit 25

Flux = $\iint \left(-P\ \frac{\partial z}{dx} - Q\ \frac{\partial z}{dy} + R\right) \cdot dA$

and replace **F** with the curl of **F**:

curl **F** = $(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\hat{\boldsymbol{\imath}} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\boldsymbol{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\hat{\boldsymbol{k}}$

So that $\iint (curl\ \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint \left(-(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\frac{\partial z}{dx} - (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\frac{\partial z}{dy} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\right) \cdot dA$

Leave that for now and begin with:

$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \int (P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \cdot dt$

$= \int (P\frac{dx}{dt} + Q\frac{dy}{dt} + R(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}) \cdot dt$

$= \int \left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + (Q + R\frac{\partial z}{\partial y})\frac{dy}{dt} \cdot dt$

$= \int \left(P + R\frac{\partial z}{\partial x}\right)dx + (Q + R\frac{\partial z}{\partial y})dy$

$= \iint \frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}(P + R\frac{\partial z}{\partial x}) \cdot dA$ by Green's theorem

where P, Q, and R are functions of x, y, and z and where z is a function of x and y:

$= \iint \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial}{\partial x}\frac{\partial z}{\partial y}\right) - (\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial}{\partial z}\frac{\partial z}{\partial x}) \cdot dA$

which reduces to

$\iint \left(-(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\frac{\partial z}{dx} - (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\frac{\partial z}{dy} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\right) \cdot dA$

which was also

$$\iint (curl\ \boldsymbol{F}) \cdot d\boldsymbol{S}$$

Therefore, $\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \iint (curl\ \boldsymbol{F}) \cdot d\boldsymbol{S}$