Divergence Theorem

In unit 23, we had a form of Green's theorem:

$\oint \mathbf{F} \cdot \mathbf{n} \cdot ds = \iint \text{div}(\mathbf{F}) \cdot dA$

where div(**F**) is the *divergence* of the field, $\mathbf{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ such that:

$\text{div}\ \mathbf{F} = \nabla \bullet \mathbf{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$

The divergence theorem is something like an extension of this where the left side becomes a two dimensional surface integral and the right side becomes a three dimensional volumetric integral:

$$\iint \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint (div\ \boldsymbol{F}) \cdot dV$$

As per usual, I will leave the proof for an appendix at the end.

Example: Find the flux of the vector field $\mathbf{F}(x,y,z) = z\cdot\hat{\imath} + y\cdot\hat{\jmath} + x\cdot\hat{k}$ across the surface of the sphere $x^2 + y^2 + z^2 = 1$.

We are asked to find the left side of the divergence theorem, so let's solve the right side.

$\text{div}\ \mathbf{F} = \nabla \bullet \mathbf{F} = \left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z}\right) = 1$

$\iiint (1) \cdot dV = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$ because the sphere has a radius of 1.

Example: Find the flux of the vector field $\mathbf{F}(x,y,z) = xy\cdot\hat{\imath} + yz\cdot\hat{\jmath} + xz\cdot\hat{k}$ across the surface of cube in the first octant bounded by x = y = z = 1.

$\text{div}\ \mathbf{F} = \nabla \bullet \mathbf{F} = \left(\frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial y} + \frac{\partial xz}{\partial z}\right) = y + z + x$

$\int_0^1 \int_0^1 \int_0^1 (y + z + x) \cdot dx \cdot dy \cdot dz = \frac{3}{2}$

Lastly, the divergence theorem gives us some sense of what the *divergence* operator means conceptually. If we integrate the divergence in a very small volume around some point and end with a positive result, then the flux out of that point is also positive, suggesting the field *diverges* from the point. If the integral of the divergence is negative, the flux is also negative, suggesting a *convergence* of the field towards that point. In fluid mechanics, such positive flux is called a *source* and such negative flux is called a *sink*.

Appendix:

Starting with the left side of the divergence theorem,

$\iint \boldsymbol{F} \cdot d\boldsymbol{S} = \iint \boldsymbol{F} \cdot \boldsymbol{n} \cdot dS = \iint \boldsymbol{P} \cdot \boldsymbol{i} \cdot \boldsymbol{n} \cdot dS + \iint \boldsymbol{Q} \cdot \boldsymbol{j} \cdot \boldsymbol{n} \cdot dS + \iint \boldsymbol{R} \cdot \boldsymbol{k} \cdot \boldsymbol{n} \cdot dS$

for the function **F** = P$\hat{\boldsymbol{\imath}}$ + Q$\hat{\boldsymbol{\jmath}}$ + R$\widehat{\boldsymbol{k}}$

We can take these one at a time:

$\iint \boldsymbol{R} \cdot \boldsymbol{k} \cdot \boldsymbol{n} \cdot dS = +\iint R(x, y, u_2(x, y)) \cdot dA$ through the top of the volume

$\iint \boldsymbol{R} \cdot \boldsymbol{k} \cdot \boldsymbol{n} \cdot dS = - \iint R(x, y, u_1(x, y)) \cdot dA$ through the bottom of the volume

$\iint \boldsymbol{R} \cdot \boldsymbol{k} \cdot \boldsymbol{n} \cdot dS = 0$ through the side of the volume

So that altogether, $\iint \boldsymbol{R} \cdot \boldsymbol{k} \cdot \boldsymbol{n} \cdot dS = \iint R(x, y, u_2(x, y)) \cdot dA - \iint R(x, y, u_1(x, y)) \cdot dA$

which is also $\iint [\int_{u_1}^{u_2} \frac{\partial R}{\partial z}(x, y, z) \cdot dz] \cdot dA = \iiint \frac{\partial R}{\partial z} \cdot dV$

Repeating for the x and y dimensions shows that the sum of the left side terms of the divergence theorem is equal to the sum of the right side terms.