Gauss's Laws

Coulomb's law gives us that the electric field from a point charges follows the equation:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R^2}\cdot\boldsymbol{\hat{R}}$$

If we imagine such a charge surrounded by a spherical surface, the flux of the electric field through that surface is then

$$\oiint\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R^2}\cdot\boldsymbol{\hat{r}}\right)\cdot d\boldsymbol{S} =$$

$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R^2}\oiint dS =$$

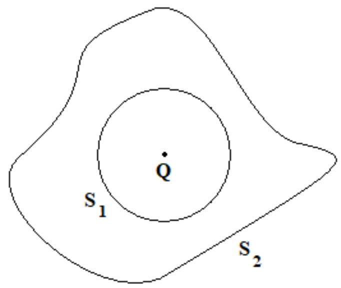
$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R^2}\cdot 4\pi R^2 =$$

$$\frac{Q}{\varepsilon_0}$$

Giving us the integral form of Gauss's law for electricity,

$$\mathrm{Q} = \varepsilon_0\cdot\oiint\boldsymbol{E}\cdot d\boldsymbol{S}$$

Vector calculus can now show that this equation is true when the charge is surrounded by a Gaussian surface of *any* shape:



Suppose we have a charge, Q, surrounded by two Gaussian surfaces, S_1 and S_2, where S_1 is a sphere and S_2 is some random closed surface. Taking the volume between the two surfaces, the divergence theorem is:

$$\iint\boldsymbol{E}\cdot d\boldsymbol{S} = \iiint(div\ \boldsymbol{E})\cdot dV$$

where the left side is two surface integrals, inward at S_1 and outward at S_2, so that

$$\iint_{S_1} \boldsymbol{E} \cdot -\boldsymbol{n} \cdot dS + \iint_{S_2} \boldsymbol{E} \cdot \boldsymbol{n} \cdot dS = \iiint (div\ \boldsymbol{E}) \cdot dV$$

If we can show that $(div\ \boldsymbol{E})$ for the electric field is zero, then

$$\iint_{S_1} \boldsymbol{E} \cdot \boldsymbol{n} \cdot dS = \iint_{S_2} \boldsymbol{E} \cdot \boldsymbol{n} \cdot dS$$

indicating the flux through the S_1 is the same as the flux through S_2. We've proved the flux through S_1 is $\frac{Q}{\varepsilon_0}$ and so that is also the flux through S_2.

To show that div **E** = 0, we take the electric field as **E** = $\frac{Q}{4\pi\varepsilon_0} \cdot \frac{\boldsymbol{R}}{|R^3|}$ and, ignoring the constants, write it as:

$$\mathbf{E} = \frac{x \cdot \hat{\boldsymbol{\imath}}}{\sqrt{x^2+y^2+z^2}^3} + \frac{y \cdot \hat{\boldsymbol{\jmath}}}{\sqrt{x^2+y^2+z^2}^3} + \frac{z \cdot \hat{\boldsymbol{k}}}{\sqrt{x^2+y^2+z^2}^3}$$

Therefore, div **E** = $\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}^3}\right)$

$$= \frac{1 - 3x^2(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{3/2}} + \frac{1 - 3y^2(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{3/2}} + \frac{1 - 3z^2(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{3/2}}$$

= 0

If we begin with the integral form of Gauss's law:

$$Q_{enc} = \varepsilon_0 \cdot \oiint \boldsymbol{E} \cdot d\boldsymbol{S}$$

$$\oiint \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{\boldsymbol{Q_{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \cdot \iiint dq = \frac{1}{\varepsilon_0} \cdot \iiint \rho \cdot dV$$

By the divergence theorem, $\oiint \boldsymbol{E} \cdot d\boldsymbol{S} = \iiint (div\ \boldsymbol{E}) \cdot dV$

so that $\iiint (div\ \boldsymbol{E}) \cdot dV = \frac{1}{\varepsilon_0} \cdot \iiint \rho \cdot dV$

leading to

$$\text{div } \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

This is known as the differential form of Gauss's law for electricity.

We can do something similar in translating between the two forms of Gauss's law for magnetism.

In the integral form, we have:

$$\oiint \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

By the divergence theorem, $\iiint (div\ \boldsymbol{B}) \cdot dV = 0$ within that closed surface

or just $\text{div}\ \mathbf{B} = 0$