Ampere – Maxwell Law

In integral form, Ampere's law is:

$$\oint \boldsymbol{B} \bullet d\boldsymbol{r} = \mu_0 I_{enclosed}$$

If we set the current through some surface equal to the current density times the normal vector of that surface, **J** • $\hat{\boldsymbol{n}}$, we can rewrite Ampere's law as:

$$\oint \boldsymbol{B} \bullet d\boldsymbol{r} = \mu_0 \iint \mathbf{J} \bullet \hat{\mathbf{n}} \cdot \mathrm{dS}$$

By Stokes' theorem, we can replace the left-hand side of this equation with:

$$\iint curl\ \boldsymbol{B} \cdot d\boldsymbol{S} \qquad \text{or} \qquad \iint curl\ \boldsymbol{B} \bullet \hat{\mathbf{n}} \cdot dS$$

Leading to

$$\text{curl } \mathbf{B} = \mu_0 \cdot \boldsymbol{J}$$

which is Ampere's law in differential form.

Now suppose we have a system where various concentrations of electric charge are moving about space as time passes. We define volumetric charge density as $\rho = \frac{Q}{V}$ making $dQ = \rho \cdot dV$.

Therefore, $\frac{dQ}{dt} = \frac{d}{dt} \iiint \rho \cdot dV = \iiint \frac{\partial \rho}{\partial t} \cdot dV$

And above, we just used $\frac{dQ}{dt} = \iint \mathbf{J} \bullet \hat{\mathbf{n}} \cdot \mathrm{dS}$

Setting them equal to each other gives us $\iiint \frac{\partial \rho}{\partial t} \cdot dV = -\oiint \mathbf{J} \bullet \hat{\mathbf{n}} \cdot \mathrm{dS}$

Negative because if there positive current flow out of the surface, $\frac{\partial \rho}{\partial t}$ is negative.

And the right side, by the divergence theorem, is $-\iiint (div\ \boldsymbol{J}) \cdot dV$

So that $\frac{\partial \rho}{\partial t} = -div\ \boldsymbol{J}$

The problem is that we saw, back in unit 23, that div (curl **B)** = 0 is always true.

But now we have $-\text{div}\,\mathbf{J} = \frac{\partial \rho}{\partial t}$ with curl **B** = $\mu_0 \cdot \boldsymbol{J}$ so that div (curl $\boldsymbol{B}$) = $-\mu_0 \frac{\partial \rho}{\partial t}$

We need to add a term involving $\frac{\partial \rho}{\partial t}$ to the right side of curl **B** = $\mu_0 \cdot \boldsymbol{J}$ so that div (curl **B)** = 0.

Let's try using Gauss's law for electricity in the differential form, $\rho = \varepsilon_0 \cdot div\boldsymbol{E}$.

$\frac{\partial \rho}{\partial t} = \varepsilon_0 \cdot div \frac{\partial \boldsymbol{E}}{\partial t}$ which gives us, if $-\text{div}\,\mathbf{J} = \frac{\partial \rho}{\partial t}$

the result div **J** + $\varepsilon_0 \cdot \text{div} \frac{\partial \boldsymbol{E}}{\partial t} = 0$ which is simplified to div $(\mathbf{J} + \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}) = 0$

With this addition to Ampere's law by Maxwell, the full Ampere-Maxwell law is then:

$$\text{curl}\,\mathbf{B} = \mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}$$

In summary, we have what are called Maxwell's equations:

Gauss's law for electricity: $\text{div}\,\mathbf{E} = \frac{\rho}{\varepsilon_0}$

Gauss's law for magnetism: $\text{div}\,\mathbf{B} = 0$

Faraday's law of induction: $\text{curl}\,\mathbf{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}$

Ampere-Maxwell law: $\text{curl}\,\mathbf{B} = \mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}$