The Speed of Light

To derive the speed of light, it is first useful to introduce an operator known as the Laplace operator or the *Laplacian*:

For $\mathbf{F} = P\cdot\boldsymbol{\hat{\imath}} + R\cdot\boldsymbol{\hat{\jmath}} + Q\cdot\boldsymbol{\hat{k}}$ the Laplacian is $\nabla^2\mathbf{F} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial z^2}$

It is also useful to know that waves travel through space according to the wave equation:

$\nabla^2\mathbf{F} = \frac{1}{\boldsymbol{v^2}} \cdot \frac{\partial^2 \boldsymbol{F}}{\partial t^2}$ where v is the speed at which the wave propagates through space

The equation relates how the field changes with respect to position in space (left side) to how the field changes with the passage of time (right side). You may have seen the same equation restricted to one-dimension as: $\frac{\partial^2 \boldsymbol{y}}{\partial x^2} = \frac{1}{\boldsymbol{v^2}} \cdot \frac{\partial^2 \boldsymbol{y}}{\partial t^2}$.

For example, if a wave has the equation y = Acos(kx – ωt) where $k = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi}{T}$, it is easy to show by taking first and then second partial derivatives that $k^2 = \frac{\omega^2}{v^2}$ which is also $v = \frac{\lambda}{T} = \lambda \cdot v$.

Beginning with the differential form of Faraday's law, curl $\mathbf{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}$, we take the curl of both sides, so that:

curl (curl $\mathbf{E}$) = curl $(-\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}})$ or $\vec{\nabla}\ x\ (\vec{\nabla}\ x\ \boldsymbol{E}) = \vec{\nabla}\ x\ (-\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}) = -\frac{\partial(\vec{\nabla}\ x\ \boldsymbol{B})}{\partial \boldsymbol{t}}$

It is also true that the curl of the curl of a field is equal to the gradient of the divergence of the field minus the Laplacian of the field.

$\vec{\nabla}\ x\ (\vec{\nabla}\ x\ \boldsymbol{E}) = \vec{\nabla}(\vec{\nabla}\bullet\boldsymbol{E}) - \nabla^2\mathbf{E}$

Therefore, we have $\vec{\nabla}\ x\ (\vec{\nabla}\ x\ \boldsymbol{E}) = \vec{\nabla}(\vec{\nabla}\bullet\boldsymbol{E}) - \nabla^2\mathbf{E} = -\frac{\partial(\vec{\nabla}\ x\ \boldsymbol{B})}{\partial \boldsymbol{t}}$

From the Ampere-Maxwell law, we know $(\vec{\nabla}\ x\ \boldsymbol{B})$ = curl $\mathbf{B} = \mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}$

giving us $\vec{\nabla}\ x\ (\vec{\nabla}\ x\ \boldsymbol{E}) = \vec{\nabla}(\vec{\nabla}\bullet\boldsymbol{E}) - \nabla^2\mathbf{E} = -\frac{\partial(\mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t})}{\partial \boldsymbol{t}}$

And by Gauss's law for electricity $\vec{\nabla}\bullet\boldsymbol{E} = \frac{\boldsymbol{\rho}}{\boldsymbol{\varepsilon_0}}$ that substitution gives us

$\vec{\nabla}(\frac{\rho}{\varepsilon_0}) - \nabla^2\mathbf{E} = -\mu_0 \frac{\partial \boldsymbol{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$

In free space, static and mobile charges are absent, so ρ and **J** are zero, reducing the above to:

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

which has the form of the wave equation where $\frac{1}{v^2} = \mu_0\varepsilon_0$

$\mu_0 = 4\pi \times 10^{-7} \frac{m \cdot kg}{s^2 \cdot A^2}$ and $\varepsilon_0 = 8.854 \times 10^{-12} \frac{s^4 \cdot A^2}{m^3 \cdot kg}$

so electromagnetic radiation propagates through empty space at a speed of

$v = 2.998 \times 10^8$ m/s

known experimentally to be the speed of light.

We could instead take the curl of both sides of the Ampere-Maxwell law so that

$$\vec{\nabla}\, x\, (\vec{\nabla}\, x\, \boldsymbol{B}) = \vec{\nabla}\, x\, (\mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}) = \vec{\nabla}\, x\, \mu_0 \cdot \boldsymbol{J} + \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \vec{\nabla}\, x\, \boldsymbol{E}}{\partial t}$$

Where, by Faraday's law, $\frac{\partial \vec{\nabla}\, x\, \boldsymbol{E}}{\partial t} = -\frac{\partial \boldsymbol{B}}{\partial t}$

Again, by the vector operator identity, $\vec{\nabla}\, x\, (\vec{\nabla}\, x\, \boldsymbol{B}) = \vec{\nabla}(\vec{\nabla} \bullet \boldsymbol{B}) - \nabla^2\mathbf{B}$

$$\vec{\nabla}(\vec{\nabla} \bullet \boldsymbol{B}) - \nabla^2\mathbf{B} = \vec{\nabla}\, x\, \mu_0 \cdot \boldsymbol{J} - \mu_0 \cdot \varepsilon_0 \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

The divergence of the magnetic field is zero (Gauss's law for magnetism) and, in free space, **J** is zero, reducing the equation to:

$$\nabla^2\mathbf{B} = \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

and yielding the same speed of light as before.