First-order Linear Differential Equations

We should begin this brief introduction with some definitions. A *differential equation* is an equation that includes one or more derivatives of some unknown function.

$\frac{dy}{dx} = 5$ is a differential equation whose solution is $y = 5x + C$

C is an arbitrary constant, so y = 5x + C is a *general solution* while y = 5x + 10 is a *particular solution*.

Another differential equation is $\frac{d^2y}{dt^2} = -9.8$ whose solution is $y = -4.9t^2 + C_1 \cdot t + C_2$

An *ordinary differential equation* is a differential equation involving derivatives with respect to one variable. The two examples above are ordinary differential equations. If derivatives with respect to more than one variable are involved, it is a *partial differential equation*. You just seen that with the 1D wave equation with involves a partial derivative with respect to time and a partial derivative with respect to horizontal position: $\frac{\partial^2 \boldsymbol{y}}{\partial x^2} = \frac{1}{\boldsymbol{v^2}} \cdot \frac{\partial^2 \boldsymbol{y}}{\partial t^2}$.

The *order* of the differential equation is designated by the highest order derivative in the equation. Therefore,

$\frac{dy}{dx} - 5 = 0$ is a *first-order* ordinary differential equation

$\frac{d^2y}{dt^2} - 9.8 = 0$ is a *second-order* ordinary differential equation

$v^2 \cdot \frac{\partial^2 \boldsymbol{y}}{\partial x^2} - \frac{\partial^2 \boldsymbol{y}}{\partial t^2} = 0$ is a *second-order* partial differential equation

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A differential equation is *linear* if it involves only first-order powers of that function multiplied by constants. A differential equation is *non-linear* if it involves powers other than one or if a derivative is multiplied by some non-constant in the function.

$\frac{dy}{dx} + 20y - 12 = 0$ is a first-order, linear differential equation

$\frac{d^2y}{dt^2} + 20y - 12 = 0$ is a second-order, linear differential equation

$\frac{d^2y}{dt^2} + 20y^2 - 12 = 0$ is a second-order, non-linear differential equation

$\frac{dy}{dx} + 20xy - 12 = 0$ is a first-order, non-linear differential equation because 20xy is not linear

Lastly, we can distinguish between *homogenous* and *non-homogenous* differential equations. If we are differentiating with respect to time, the differential equation is non-homogenous if there is some term only dependent upon time. The differential equation is homogenous if there is no such term.

$\frac{dy}{dt} + 20y - 12 = 0$ is a homogenous differential equation

$\frac{dy}{dt} + 20t - 12 = 0$ is a non-homogenous differential equation

$\frac{dy}{dt} + 20yt - 12 = 0$ is a homogenous differential equation; $20yt$ is not solely a function of time

You can think of the non-homogenous term as some outside influence on the system that does not depend on the system itself.

To begin, let's just practice testing solutions for differential equations.

Example: Determine whether or not the following functions are solutions to the differential equation $xy' - 2y = x^3e^x$.

A. $y = x^2$
B. $y = x^2e^x$
C. $y = x^2(2 + e^x)$

Making the substitutions for A, we have

$x \cdot 2x - 2(x^2) = x^3 \cdot e^x$

$0 = x^3 \cdot e^x$ shows A is not a solution

In B, we have

$x \cdot (x^2e^x + 2xe^x) - 2(x^2e^x) = x^3e^x$

$x^3e^x = x^3e^x$ shows B is a solution

In C, we have

$x \cdot (x^2e^x + 4x + 2xe^x) - 2(2x^2 + x^2e^x) = x^3e^x$

$x^3e^x = x^3e^x$ shows C is a solution

Differential equations in the form of M(x) + N(y)·$\frac{dy}{dx}$ = 0 can be solved with separation of variables.

Example: Solve the differential equation $\frac{dy}{dx} = \frac{6-x^2}{2y^3}$.

$2y^3 \cdot dy = (6 - x^2) \cdot dx$

$$\int 2y^3 \cdot dy = \int (6 - x^2) \cdot dx$$

$$\frac{1}{2}y^4 = 6x - \frac{1}{3}x^3 + C$$

First-order linear differential equations in the form of $\frac{dy}{dx}$ + P(x)·y = Q(x) can be solved with an integrating factor, u(x) = $e^{\int P(x)dx}$.

Example: Solve the differential equation y' + 2xy = 10x.

Looking at the general form above, we can see that P(x) = 2x. The integrating factor is then u(x) = $e^{\int (2x)dx} = e^{x^2}$

Multiply all terms in the differential equation by the integrating factor so that:

y' + 2xy = 10x

becomes

$e^{x^2} \cdot (\frac{dy}{dx}) + e^{x^2} \cdot 2xy = e^{x^2} \cdot 10x$

then integrate both sides

$$\int \left[e^{x^2} \cdot \left(\frac{dy}{dx}\right) + e^{x^2} \cdot 2xy\right] dx = \int \left(e^{x^2} \cdot 10x\right) dx$$

Note that the left side, by the product rule is equal to y·e^{x^2}

$y \cdot e^{x^2} = \int \left(e^{x^2} \cdot 10x\right) dx = 5 \cdot e^{x^2} + C$ using substitution

$y = 5 + Ce^{-x^2}$

You can verify the solution by putting this result into the original differential equation.