First-order Linear ODEs – Applications

Example: In a room with a constant temperature of 26ºC, an iron pan is found to cool from 180ºC to 100º in five minutes. What will be the temperature after five more minutes, assuming Newton's law of cooling: $\frac{dT}{dt} = -kT$?

First, we solve the differential equation with separation of variables.

$$\frac{dT}{dt} = -kT$$

$$\frac{1}{T}dT = -k \cdot dt$$

$$\int_{T_i}^{T_f} \frac{1}{T}dT = \int_0^t -k \cdot dt$$

$$\ln\left(\frac{T_f}{T_i}\right) = -kt$$

$$T_f = T_i \cdot e^{-kT}$$

Putting in the values given, we have $100 = 180 \cdot e^{-k \cdot 5}$ so $k = 0.118 \frac{1}{min}$

Using the same derived equation, we have $T_f = 180 \cdot e^{-(0.118)(10)}$ and $T_f = 55.3$ºC

Example: The population, *P*, of a certain species is believed to change at the rate

$$\frac{dP}{dt} = kP(1 - \frac{P}{4000})$$

In five years, the population has grown from 40 initial individuals to 104. How many individuals will there be after ten more years?

This is a specific type of exponential relationship called a *logistical differential equation*. It is very similar to exponential growth, but instead has some maximum value called the *carrying capacity* of the system. Here, that value is 4000 because when P = 4000, $\frac{dP}{dt} = 0$.

As before with the first example, solve the differential equation to find *k*.

$$\frac{dP}{dt} = kP(1 - \frac{P}{4000})$$

$$\frac{1}{P(1-\frac{P}{4000})}\, dP = k \cdot dt$$

$$\left(\frac{1}{P} + \frac{1}{4000-P}\right) dP = k \cdot dt$$

$$\int \left(\frac{1}{P} + \frac{1}{4000-P}\right) dP = \int k \cdot dt$$

ln|P| - ln|4000 – P| = k·t + C

$$\frac{4000-P}{P} = e^{-kt-C} = e^{-C}e^{-kt} = b \cdot e^{-kt}$$

$$P = \frac{4000}{1+b \cdot e^{-kt}}$$

We know that when t = 0, P = 40, so we can solve for b = 99

Then, solving again we have $104 = \frac{4000}{1+99 \cdot e^{-k \cdot 5}}$ and k = 0.194 $\frac{1}{years}$

Then, solving one last time, P = $\frac{4000}{1+99 \cdot e^{-(0.194)(15)}}$ = 626 individuals.

Example: A circuit contains an ideal battery with voltage, *V*, an ideal resistor with resistance, *R*, an ideal inductor with inductance, *L*, and an open switch. When the switch is closed, the battery drives a current, *I*, through the loop in which the voltage across the resistor is -*IR* and the voltage across the inductor is $-L\frac{dI}{dt}$. Using Kirchhoff's loop rule, determine the current as a function of time.

We begin with the loop rule, 0 = $V - IR - L\frac{dI}{dt}$

This takes the form of a first-order linear differential equation, $\frac{dy}{dx}$ + P(x)·y = Q(x)

$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$ which we multiply by the integrating factor $e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

$$e^{\frac{Rt}{L}}(\frac{dI}{dt}) + e^{\frac{Rt}{L}}(\frac{R}{L}I) = e^{\frac{Rt}{L}}(\frac{V}{L})$$

$$\int [e^{\frac{Rt}{L}}\left(\frac{dI}{dt}\right) + e^{\frac{Rt}{L}}\left(\frac{R}{L}I\right)]dt = \int e^{\frac{Rt}{L}}\left(\frac{V}{R}\right)\mathrm{dt}$$

$$\mathrm{I}\cdot e^{\frac{Rt}{L}} = \frac{V}{R}\cdot(e^{\frac{Rt}{L}} - 1)$$

$$\mathrm{I} = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

This agrees with an initial current of zero when t = 0 and $\lim_{t\to\infty} I = \frac{V}{R}$.