Second-order Linear Differential Equations

These equations have the form $P(x)\cdot\frac{d^2y}{dx^2} + Q(x)\cdot\frac{dy}{dx} + R(x)\cdot y = G(x)$

If $G(x) = 0$, the equation is homogenous; if $G(x) \neq 0$, the equation is non-homogenous.

For homogenous equations, it is fairly easy to show that $y_1(x)$ is a solution and $y_2(x)$ is a solution, then $c_1\cdot y_1(x) + c_2\cdot y_2(x)$ is also a solution, where c_1 and c_2 are constants. For instance, suppose $y_3(x) = c_1\cdot y_1(x) + c_2\cdot y_2(x)$ where y_1 and y_2 are known solutions.

$P(x)y_3'' + Q(x)y_3' + R(x)y_3 = 0$

$P(x)(c_1\cdot y_1 + c_2\cdot y_2)'' + Q(x)(c_1\cdot y_1 + c_2\cdot y_2)' + R(x)(c_1\cdot y_1 + c_2\cdot y_2) = 0$

$P(x)(c_1\cdot y_1)'' + P(x)(c_2\cdot y_2)'' + Q(x)(c_1\cdot y_1)' + Q(x)(c_2\cdot y_2)' + R(x)(c_1\cdot y_1) + R(x)(c_2\cdot y_2) = 0$

Because, for such linear equations, the derivative of the sum is the sum of the derivatives.

$[P(x)(c_1\cdot y_1)'' + Q(x)(c_1\cdot y_1)' + R(x)(c_1\cdot y_1)] + [P(x)(c_2\cdot y_2)'' + Q(x)(c_2\cdot y_2)' + R(x)(c_2\cdot y_2)] = 0$

$c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] = 0$

$c_1\cdot 0 + c_2\cdot 0 = 0$ which is true, which validates our assumption of y_3 as a solution

If y_1 and y_2 are linearly independent equations, then y_3 is the general solution to the differential equation.

Let's try solving a relatively simple homogenous equation where P(x), Q(x) and R(x) are all constants, so that:

$a\cdot\frac{d^2y}{dx^2} + b\cdot\frac{dy}{dx} + c\cdot y = 0$ otherwise written as $a\cdot y'' + b\cdot y' + c\cdot y = 0$

We're looking for some kind of function where all three terms are similar enough that they can be set to zero to find the values of a, b, and c. We know that exponential functions are similar to their derivatives, so let's make a guess that the solution will be $y = e^{rx}$.

If $y = e^{rx}$ then $y' = r\cdot e^{rx}$ and $y'' = r^2\cdot e^{rx}$

$a\cdot y'' + b\cdot y' + c\cdot y = 0$ becomes $a\cdot r^2\cdot e^{rx} + b\cdot r\cdot e^{rx} + c\cdot e^{rx} = 0$

which can be simplified to $(a\cdot r^2 + b\cdot r + c)\cdot e^{rx} = 0$

$e^{rx} = 0$ is a trivial solution, leaving only the possibility that $a\cdot r^2 + b\cdot r + c = 0$. This is called the *auxiliary equation* and is solved with the quadratic formula:

$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ or, divided: $r_1 = \frac{-b + \sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$

There are three possibilities when solved in this way:

$\sqrt{b^2 - 4ac}$ is greater than zero

$\sqrt{b^2 - 4ac}$ is zero

$\sqrt{b^2 - 4ac}$ is less than zero

We'll take each in turn.

If $\sqrt{b^2 - 4ac}$ is greater than zero, we simply have two real and distinct solutions:

y = $e^{r_1 x}$ and y = $e^{r_2 x}$

So, by linear combination, the general solution is:

y = $c_1 \cdot e^{r_1 x} + c_2 \cdot e^{r_2 x}$

If $\sqrt{b^2 - 4ac}$ is zero, r = $\frac{-b}{2a}$ and y = e^{rx} is a solution

It can also be shown that y = $x \cdot e^{rx}$ is a second solution

$y'' = xr^2 \cdot e^{rx} + 2r \cdot e^{rx}$ $y' = x \cdot re^{rx} + e^{rx}$ $y = x \cdot e^{rx}$

$a \cdot y'' + b \cdot y' + c \cdot y = 0$ becomes $a \cdot (xr^2 \cdot e^{rx} + 2r \cdot e^{rx}) + b \cdot (x \cdot re^{rx} + e^{rx}) + c \cdot (xe^{rx}) = 0$

combining terms, we have $(ar^2 + br + c)xe^{rx} + (2ar + b)e^{rx} = 0$

because $ar^2 + br + c = 0$ is our initial auxiliary equation and $2a(\frac{-b}{2a}) + b = 0$.

By linear combination, the general solution is then y = $c_1 \cdot e^{rx} + c_2 \cdot xe^{rx}$

If $\sqrt{b^2 - 4ac}$ is less than zero, then r_1 and r_2 have imaginary components and must be written as complex numbers.

$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$

For compactness, we will say $\alpha = \frac{-b}{2a}$ and $\beta = \frac{-\sqrt{b^2-4ac}}{2a}$

So that $r_1 = \alpha + i \cdot \beta$ and $r_2 = \alpha - i \cdot \beta$

By linear combination, the general solution is then y = $C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$

We can rewrite this as $y = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$

By Euler's equation, $e^{i\theta} = \cos(\theta) + i\cdot\sin(\theta)$

We can replace $e^{i\beta x}$ with $\cos(\beta x) + i\cdot\sin(\beta x)$

giving us $y = C_1 e^{\alpha x} \cdot[\cos(\beta x) + i\cdot\sin(\beta x)] + C_2 e^{\alpha x} \cdot[\cos(\beta x) - i\cdot\sin(\beta x)]$

$y = e^{\alpha x}[(C_1 + C_2)\cos(\beta x) + i(C_1 - C_2)\sin(\beta x)]$

$y = e^{\alpha x}\cdot[c_1\cdot\cos(\beta x) + c_2\sin(\beta x)]$

The values for c_1 and c_2 depend upon the initial values for y and y'.

Example: Solve the equation: y" + y' – 6y = 0 given initial values y(0) = 1 and y'(0) = 0.

First, we find the auxiliary equation to be $1\cdot r^2 + 1\cdot r - 6r = 0$.

Solving for r (here factorable), we get $r_1 = 2$ and $r_2 = -3$

By linear combination, we have $y = c_1\cdot e^{2x} + c_2\cdot e^{-3x} = 1$.

Differentiating, we have $y' = 2c_1\cdot e^{2x} - 3c_2\cdot e^{-3x} = 0$.

Now we have two equations and two unknowns, given that x = 0.

$1 = c_1 + c_2$ and $0 = 2c_1 - 3c_2$

$c_1 = \frac{3}{5}$ and $c_2 = \frac{2}{5}$

making the solution $y = \frac{3}{5}\cdot e^{2x} + \frac{2}{5}\cdot e^{-3x}$

If the differential equation is non-homogenous, it takes the form

$$P(x)\cdot\frac{d^2y}{dx^2} + Q(x)\cdot\frac{dy}{dx} + R(x)\cdot y = G(x)$$

The solution to the non-homogenous equation is related to its complimentary homogenous equation

$$P(x)\cdot\frac{d^2y}{dx^2} + Q(x)\cdot\frac{dy}{dx} + R(x)\cdot y = 0$$

If we call y_p a particular solution to the non-homogenous equation and y_c the general solution to the homogenous form, we can show that the general solution to the non-homogenous equation is:

$y = y_p + y_c$

We prove this by rearranging

$y_c = y - y_p$

$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = 0$

$(ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) = 0$

$(ay'' + by' + cy) = G(x)$ as the general solution to the non-homogenous equation

$(ay_p'' + by_p' + cy_p) = G(x)$ as the solution to the particular non-homogenous equation

and $G(x) - G(x) = 0$

Knowing already how to find the solution to the homogenous equation, if we know methods to find a particular non-homogenous solution, we have enough to write the general non-homogenous solutions. There are two such methods, undetermined coefficients (easier, sometimes useful) and variation of parameters (difficult, always useful).

Example using undetermined coefficients: Solve the differential equation, $y'' + 4y = e^{3x}$.

First, solve the complimentary homogenous equation, $y'' + 4y = 0$

$1 \cdot r^2 + 0 \cdot r + 4 = 0$

$r = \pm 2i$ so $\alpha = 0$ and $\beta = 2$ gives us $y_c = c_1 \cdot \cos(2x) + c_2 \sin(2x)$

Now we look at $G(x) = e^{3x}$ and make a guess that the solution may be of similar form, $y = Ae^{3x}$. Here, A is the undetermined coefficient we're looking-for.

If $y = Ae^{3x}$ then $y' = 3Ae^{3x}$ and $y'' = 9A^{3x}$

Putting those into $y'' + 4y = e^{3x}$ and solving gives us $A = \frac{1}{13}$

making $y_p = \frac{1}{13}e^{3x}$ our particular solution

The general solution is then $y = \frac{1}{13}e^{3x} + c_1 \cdot \cos(2x) + c_2 \sin(2x)$

Again, c_1 and c_2 would depend upon whatever initial values could be given.

Generally speaking, you look at G(x) and attempt a solution that has a similar form with undetermined coefficients. However, be careful that your attempt does not solve the homogenous form because it cannot also solve the particular non-homogenous form. Try something similar with an x or an x^2. Sometimes, it's just trial and error.

To solve by variation of parameters, take the general solution to the homogenous form:

$y = c_1 \cdot y_1(x) + c_2 \cdot y_2(x)$ and replace c_1 and c_2 with functions of x, such that

$y_p = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x)$

We then use a system of equations with

$u_1'y_1 + u_2'y_2 = 0$
$u_1'y_1' + u_2'y_2' = G(x)$

Solve for u_1 and u_2 and plug them into the equation above for the solution.

Example: Solve the differential equation, $y'' + y = 1 + \tan(x)$

Solving the homogenous equation, $y'' + y = 0$, we have $y_c = c_1\cos(x) + c_2\sin(x)$

Writing this with parameters gives us

$y_p = u_1(x)\cos(x) + u_2(x)\sin(x)$

The system of equations is then

$u_1'\cos(x) + u_2'\sin(x) = 0$
$u_1' \cdot -\sin(x) + u_2' \cdot \cos(x) = G(x) = 1 + \tan(x)$

Solving gives us

$u_1' = -\sin(x)(1 + \tan(x))$

$u_2' = \cos(x)(1 + \tan(x))$

Integrating

$u_1 = \cos(x) + \sin(x) – \ln(\sec(x) + \tan(x))$

$u_2 = \sin(x) – \cos(x)$

Which are then put into $y_p = u_1(x)\cos(x) + u_2(x)\sin(x)$ so that

$y_p = [\cos(x) + \sin(x) – \ln(\sec(x) + \tan(x))](x)\cos(x) + [\sin(x) – \cos(x)](x)\sin(x)$