Second-order Linear ODEs – Applications

Example: Along a frictionless, level surface, a block of mass *m* is attached to an ideal spring with spring constant, *k*. The block is pulled right a distance, *A*, and released. Find the position of the block as a function of time, *x(t)*.

From Newton's second law, we have

ma = -kx

$a = -\frac{k}{m}x$

$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

$x'' + \frac{k}{m}x = 0$

The auxiliary equation here is $r^2 + 0 + \frac{k}{m} = 0$

$r_1 = i\sqrt{\frac{k}{m}}$ $\qquad r_2 = -i\sqrt{\frac{k}{m}}$

For $r_1 = \alpha + i\cdot\beta$ and $r_2 = \alpha - i\cdot\beta$

$\alpha = 0$ and $\beta = \sqrt{\frac{k}{m}}$

$x = e^{\alpha x}\cdot[c_1\cdot\cos(\beta x) + c_2\sin(\beta x)] = c_1\cdot\cos(\sqrt{\frac{k}{m}}t) + c_2\cdot\sin(\sqrt{\frac{k}{m}}t)$

At time zero, we have

$A = c_1\cdot\cos(0) + c_2\cdot\sin(0)$ so $c_1 = A$

If we differentiate, we have

$v_x = -\sqrt{\frac{k}{m}}\cdot c_1\cdot\sin(\sqrt{\frac{k}{m}}t) + \sqrt{\frac{k}{m}}\cdot c_2\cdot\cos(\sqrt{\frac{k}{m}}t)$

At time zero, we have

$0 = 0 + \sqrt{\frac{k}{m}}\cdot c_2\cdot\cos(0)$ so $c_2 = 0$

$x = A\cdot\cos(\sqrt{\frac{k}{m}}t)$

Example: Along a frictionless, level surface, a block of mass *m* is attached to an ideal spring with spring constant, *k*. The block is pulled right a distance, *A*, and released. The ambient air creates a fluid drag force such that $F_d = -bv$. Find the position of the block as a function of time, *x(t)*.

From Newton's second law, we have

ma = – kx – bv

a = $-\frac{k}{m}x - \frac{b}{m}v$

$\frac{d^2x}{dt^2} + \frac{b}{m} \cdot \frac{dx}{dt} + \frac{k}{m}x = 0$

x'' + $\frac{b}{m}$x' + $\frac{k}{m}x$ = 0

Here, the complimentary equation is

$r^2 + \frac{b}{m} \cdot r + \frac{k}{m} = 0$

$r_1 = \frac{-b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$ $r_2 = \frac{-b}{2m} - \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$

If we reverse the order under the radical,

$r_1 = \frac{-b}{2m} + i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ $r_2 = \frac{-b}{2m} - i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

where $\alpha = \frac{-b}{2m}$ and $\beta = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Using the form $x = e^{\alpha x} \cdot [c_1 \cdot \cos(\beta x) + c_2 \sin(\beta x)]$

We have $x = e^{\frac{-bt}{2m}} \cdot [c_1 \cdot \cos(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \cdot t) + c_2 \sin(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \cdot t)]$

Given the same initial conditions as before, $c_1 = A$ and $c_2 = 0$

$x = e^{\frac{-bt}{2m}} \cdot [A \cdot \cos(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \cdot t)]$

Note the first term is a decay function while the second term oscillates according to the physical properties of the system.

Example: Along a frictionless, level surface, a block of mass *m* is attached to an ideal spring with spring constant, *k*. The block is pulled right a distance, *A*, and released. As it oscillates, an outside force is applied sinusoidally with the function, $F = F_0 \cdot \cos(\omega \cdot t)$. Find the position of the block as a function of time, *x(t)*.

From Newton's second law, we have

$ma = - kx + F_0 \cdot \cos(\omega \cdot t)$

$a = -\frac{k}{m}x + \frac{F_0}{m} \cdot \cos(\omega \cdot t)$

$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m} \cdot \cos(\omega \cdot t)$

$x'' + \frac{k}{m}x = \frac{F_0}{m} \cdot \cos(\omega \cdot t)$

This a non-homogenous equation with $G(x) = \frac{F_0}{m} \cdot \cos(\omega \cdot t)$

We can try $x = C \cdot \cos(\omega t)$ where C is the undetermined coefficient

$-\omega^2 \cdot C \cdot \cos(\omega t) + \frac{k}{m}x = \frac{F_0}{m} \cdot \cos(\omega t)$

$C = \frac{\frac{F_0}{m}}{\frac{k}{m} - \omega^2}$ gives us $x = \frac{\frac{F_0}{m}}{\frac{k}{m} - \omega^2} \cdot \cos(\omega t)$ as our particular solution

We add this to the general solution $x = c_1 \cdot \cos(\sqrt{\frac{k}{m}}t) + c_2 \cdot \sin(\sqrt{\frac{k}{m}}t)$

to get $x = c_1 \cdot \cos(\sqrt{\frac{k}{m}}t) + c_2 \cdot \sin(\sqrt{\frac{k}{m}}t) + \frac{\frac{F_0}{m}}{\frac{k}{m} - \omega^2} \cdot \cos(\omega t)$

Given the same initial conditions as before, $c_1 = A$ and $c_2 = 0$

$x = A \cdot \cos(\sqrt{\frac{k}{m}}t) + \frac{\frac{F_0}{m}}{\frac{k}{m} - \omega^2} \cdot \cos(\omega t)$

Note that as the driving frequency, ω, approaches $\frac{k}{m}$, the second term increases. In the classic wine glass and speaker demonstration, as the driving frequency of the speaker approaches the natural frequency of the wine glass, the amplitude increases beyond the structural capacity of the glass, forcing it to shatter.