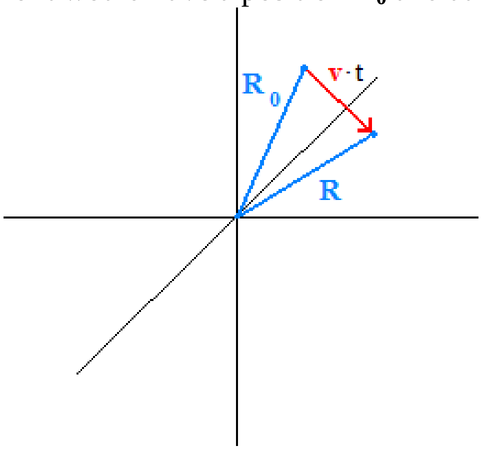
Equations of Lines and Planes

Imagine a particle moving through three-dimensional space at some constant velocity, **v**. At time zero it would have a position $\mathbf{R_0}$ and at some later time, t, it would have a position **R**.



The motion of the particle forms a line of all points: $\mathbf{R} = \mathbf{R_0} + \mathbf{v}\cdot t$ for all times $-\infty < t < \infty$ of which only a small segment above is shown in red.

If we expand **v** into its components so that $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, then we have the parametric equations:

$x = x_0 + v_x \cdot t$
$y = y_0 + v_y \cdot t$
$z = z_0 + v_z \cdot t$

For example, suppose $\mathbf{R_0} = (1, 2, 3)$ and $\mathbf{v} = \langle 4, 5, 6 \rangle$, then

$x = 1 + 4t$
$y = 2 + 5t$
$z = 3 + 6t$

At two seconds, the position would then be (9, 12, 15) and we could just as easily define the same line with

$x = 9 + 4t$
$y = 12 + 5t$
$z = 15 + 6t$

We would also have the same line in space if the particle was moving twice as fast, so that $\mathbf{v} = \langle 8, 10, 12 \rangle$ and

$x = 1 + 8t$
$y = 2 + 10t$
$z = 3 + 12t$

Lastly, we can take the three equations of a line and solve for time so that:

$x = x_0 + v_x \cdot t$
$y = y_0 + v_y \cdot t$
$z = z_0 + v_z \cdot t$

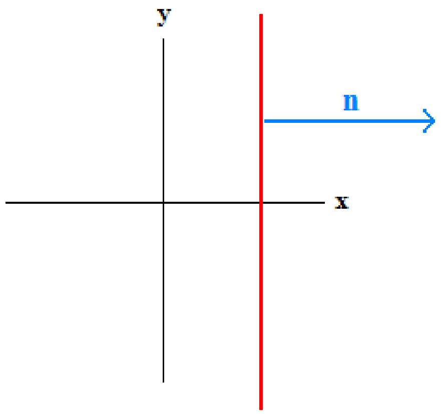
become

$$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

which are called the symmetric equations, where v_x, v_y, and v_z are called the directional numbers of the line.

A plane is a flat surface which can be defined with a point within the plane, $P_0 = (x_0, y_0, z_0)$, and a normal vector, **n**, which is orthogonal, or perpendicular, to the plane.

For a simple example, let's take a plane including the point (5, 0, 0) which is parallel to the y-z plane.



Even though it doesn't really matter, we can use a convention that the normal vector points away from the y-z plane.

If **n** = <1, 0, 0>, then the plane is defined by P = (5, 0, 0) and **n** = <1, 0, 0>.

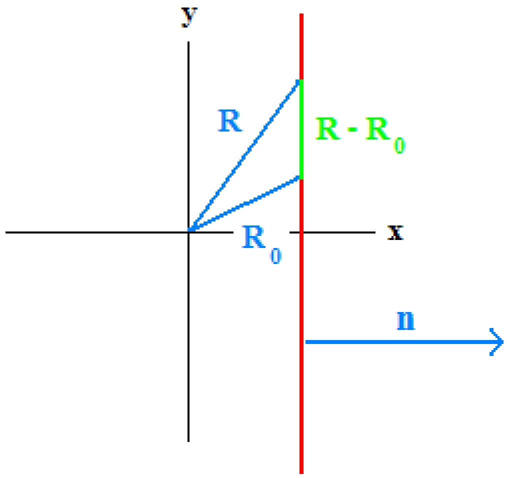
Of course, we can use any point on the plane, so this plane is defined just as well with:

P = (5, 6, -10) and **n** = <1, 0, 0>.

And it's really only the *direction* of the normal vector that is important, so this plane is also equally defined with:

P = (5, 0, 0) and **n** = <2, 0, 0> or P = (5, 6, -10) and **n** = <15, 0, 0>

Let $\mathbf{R} - \mathbf{R_0}$ be a line segment in a given plane between points P and P_0. If this line segment is *in* the plane and the normal vector, **n**, is *perpendicular* to the plane, then the dot product tells us that $\mathbf{n} \bullet (\mathbf{R} - \mathbf{R_0}) = 0$ which is often written as $\mathbf{n} \bullet \mathbf{R} = \mathbf{n} \bullet \mathbf{R_0}$ and called the vector equation of the plane.



We can see it is true in the diagram above. $\mathbf{n} \bullet \mathbf{R}$ equals |n| times the x-component of R and $\mathbf{n} \bullet \mathbf{R_0}$ equals |n| times the x-component of R_0. Those two x-components are equal so $\mathbf{n} \bullet \mathbf{R} = \mathbf{n} \bullet \mathbf{R_0}$.

If we expand the three vectors so that

$\mathbf{n} = \langle a, b, c\rangle$
$\mathbf{R} = \langle x, y, z\rangle$
$\mathbf{R_0} = \langle x_0, y_0, z_0\rangle$

then we can use $\mathbf{n} \bullet (\mathbf{R} - \mathbf{R_0}) = 0$ to see that

$\langle a, b, c\rangle \bullet \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$a\cdot(x - x_0) + b\cdot(y - y_0) + c\cdot(z - z_0) = 0$ called the scalar equation of the plane

or

$a\cdot x + b\cdot y + c\cdot z + d = 0$ if $d = -(a\cdot x_0 + b\cdot y_0 + c\cdot z_0)$

and is called the linear equation of the plane

Suppose we are given three points in space and we wish to find the plane that contains them. The points are:

P = (1, 3, 2)   Q = (3, 1, 6)   R = (5, 2, 0)

A plane is defined by a point within the plane and a normal vector. We can use any of the three as a defining point, so we need to only determine the normal vector. We also know from previous notes that the cross product between any two lines gives us a vector perpendicular to both. So the cross product of lines $\overline{PQ}$ and $\overline{PR}$ will give us the normal vector we need.

$\overline{PQ}$ = Q – P = (4, -2, 2)   and $\overline{PR}$ = R – P = (-6, 1, 2)

$$\overline{PQ} \mathbf{x} \overline{PR} = \begin{array}{ccc} i & j & k \\ 4 & -2 & 2 \\ -6 & 1 & 2 \end{array} = -6\mathbf{i} - 20\mathbf{j} - 8\mathbf{k}$$

So the plane can be defined with point P = (1, 3, 2) and normal vector **n** = -6**i** – 20**j** – 8**k**

Or in the form $a\cdot(x - x_0) + b\cdot(y - y_0) + c\cdot(z - z_0) = 0$ we have

$-6(x - 1) + -20(y - 3) + -8(z - 2) = 0$

$6x + 20y + 8z - 72 = 0$

Two planes are parallel if their normal vectors are parallel (so that $\mathbf{n_1} \propto \mathbf{n_2}$). If not, the two planes will cross at a line and form an acute angle between themselves. The angle and the line can be determined as follows:

Suppose the two planes have the linear equations x + y + z = 1 and x – 2y + 3z = 1.

The angle between the planes is the same as the angle between their normal vectors, so we can use the dot product:

$\mathbf{n_1} \bullet \mathbf{n_2} = |n_1||n_2|\cos\theta$ and solve for θ

$<1, 1, 1> \bullet <1, -2, 3> = \sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{1^2 + -2^2 + 3^2} \cdot \cos\theta$

$1 - 2 + 3 = \sqrt{3} \cdot \sqrt{14} \cdot \cos\theta$

$\theta = 72^o$

To determine the line at which they intersect, we first need to find at least one point which sits on that line.

Because neither plane is parallel to the x-y plane, the line they form when they cross must also cross the x-y plane, where z = 0. Therefore, if we use that point,

$x + y = 1$ and $x - 2y = 1$ so $x = 1$ and $y = 0$

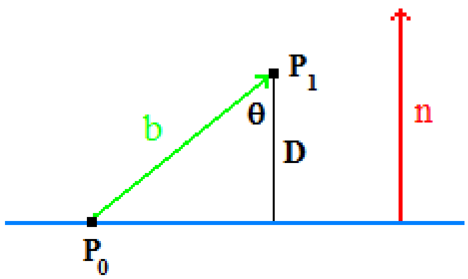
Also, if the line of intersections lies in both planes, then it must be perpendicular to both normal vectors. This is also true, as we saw before, of the cross product of the two normal vectors. So

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{array}{ccc} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{array} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

The equation of the line of intersection can then be written in the form $\mathbf{R} = \mathbf{R_0} + \mathbf{v}\cdot t$

$\mathbf{R} = (1, 0, 0) + \langle 5, -2, -3\rangle \cdot t$

Lastly, how can we determine the distance, *D*, between a point in space, $P_1 = (x_1, y_1, z_1)$, and a plane of the equation $a\cdot x + b\cdot y + c\cdot z + d = 0$?



Let **b** serve as a vector from P_0 to P_1 so that $\mathbf{b} = (x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0)$. *D* is then the scalar component of **b** along **n** (as was defined in the previous notes):

$$D = \frac{|\boldsymbol{n}\bullet\boldsymbol{b}|}{|\boldsymbol{n}|} = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2+b^2+c^2}} = \frac{(ax_1+by_1+cz_1) - (ax_0+by_0+cz_0)}{\sqrt{a^2+b^2+c^2}}$$

$$D = \frac{|a\cdot x_1 + b\cdot y_1 + c\cdot z_1 + d|}{\sqrt{a^2+b^2+c^2}}$$