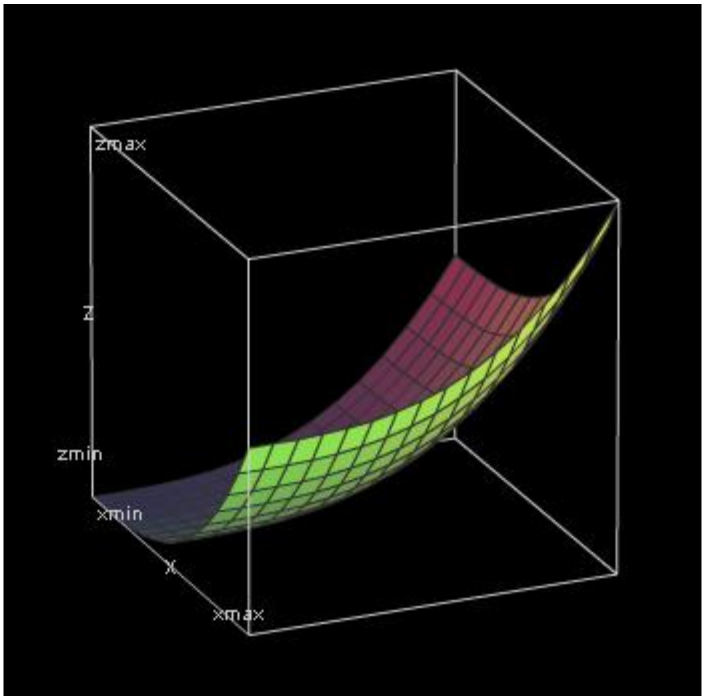
Functions of Several Variables

A function of two variables is a function that assigns a real number, *z*, to any ordered pair of real numbers, *x* and *y*, such that $z = f(x, y)$.

An example would be how atmospheric pressure at any given time depends upon the latitude and longitude coordinates of the Earth's surface. Here, the latitude and longitude are independent variables with values that exist in the domain and the atmospheric pressure is a dependent variable with values that exist in the range.

We generally graph these functions by imagining the x-y plane lying horizontal and containing the ordered pairs of the domain. Above this plane (or below), along the z-axis, we plot the corresponding values of the range.

For example, here is a graph of $z = x^3 + y^3$ in the domain $(0 < x < 1, 0 < y < 1)$ and range $(0 < z < 2)$:

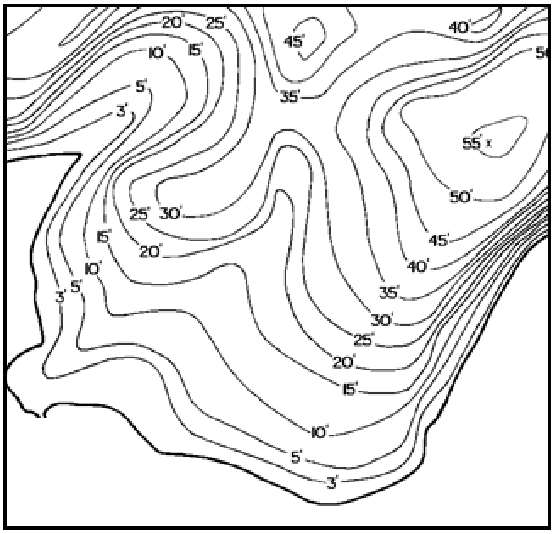


using the program at:
http://www.math.uri.edu/~bkaskosz/flashmo/graph3d/

When x = 0 and y = 0, z = 0 shown in the back-left corner.
When x = 1 and y = 1, z = 2 shown in the top-right corner.

We've already seen graphs like this in the notes on equations of cylinders and quadratic surfaces.

Another way of drawing these functions is draw them on a two-dimensions plane, but to use *level curves* that contain a constant value in the range. The most common example of this is topographical maps where every curve on the map represents a constant altitude:



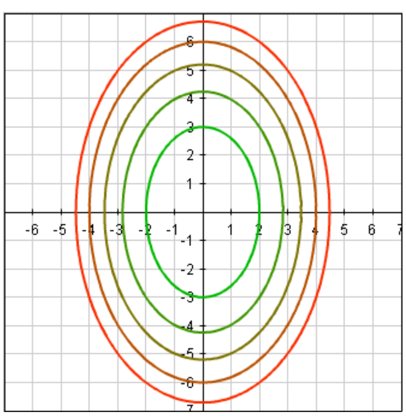
When the level curves above are far-apart it indicates one can move far in the x-y plane without a significant change in altitude, thus you are hiking along a shallow slope. When the level curves are close-together, motion in the x-y plane corresponds to a greater change in altitude, suggesting a steep slope.

Suppose we want to draw a series of level curves for the function $f(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2$.

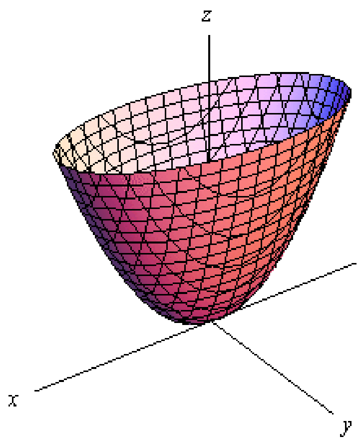
If we rewrite this as $z = \frac{x^2}{4} + \frac{y^2}{9}$ then $1 = \frac{x^2}{4z} + \frac{y^2}{9z}$

which is the equation of an ellipse with vertices of $\pm 2\sqrt{z}$ and $\pm 3\sqrt{z}$.

For a contour map, we can just select a set of values for z and draw those corresponding ellipses. Here is the contour map for z = 1, 2, 3, 4, and 5.



To visualize the corresponding three-dimensional curve, imagine taking each contour curve and elevating it along the z-axis to its respective value. Doing so for this same function would produce the elliptic paraboloid:



Functions of more than two variables are fairly easy to write with equations, i.e.

$f(x, y, z) = 2x^2 + 3y^2 + 4z^2$

but are difficult to visualize because it would require imagining shapes in four dimensions.

At best, we can visualize level surfaces, surfaces in which the range value is a constant. For example, in the equation above, we could set f(x, y, z) = 10 and find that the resulting surface is an ellipsoid. When f(x, y, z) = 12, we have a slightly larger ellipsoid enclosing the first, but still concentric to the origin.