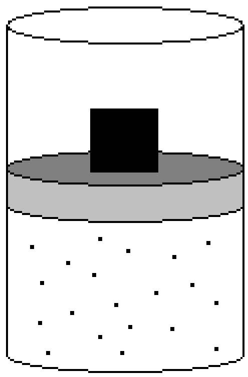
Partial Derivatives

Suppose we have an ideal gas enclosed in a chamber covered with a piston free to slide vertically:



Let's say we have 2.0 moles of gas at 300K, where the block resting on top of the piston provides a pressure of 10,000Pa. By solving the ideal gas law, PV = nRT, where R = 8.3145, we find the volume of the gas to be $0.4989m^3$.

Now suppose we place a flame under the gas so that the temperature rises from 300K to 301K. The pressure is staying constant at 10,000Pa because of the block, so we can again use the ideal gas law to find the new volume to be $0.5005m^3$.

We would like some way to notate this relationship between a change in temperature and a resultant change in volume (keeping the pressure constant).

The most common notation is: $\frac{\partial V}{\partial T}$ where the symbol ∂ indicates a *partial derivative*.

Again, this means the change in volume divided by the corresponding change in temperature (taken to be infinitely small), assuming all other variables are held constant.

We could have just as easily locked the piston in place and held a flame under the candle. In this case, the volume would stay fixed, the pressure of the gas would rise, and we could determine the value of $\frac{\partial P}{\partial T}$.

The process of computing partial derivatives is simple. Suppose we have the function:

$z = f(x, y) = 10x^2y - 12x + 15y^3 + 3yx^3$

If we want $\frac{\partial z}{\partial x}$, we are assuming that we are holding *y* constant, so we simply differentiate the function with respect to *x*, treating *y* as a constant so that:

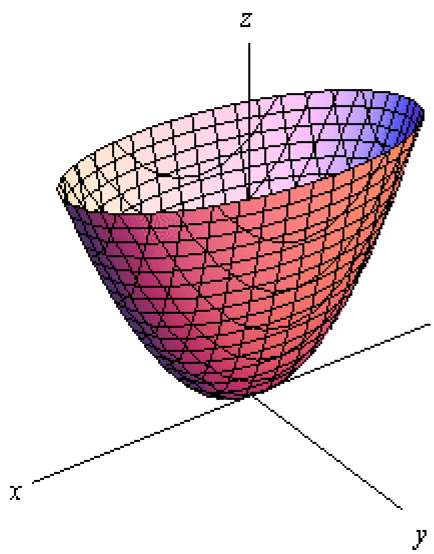
$\frac{\partial z}{\partial x} = 20xy - 12 + 0 + 9yx^2$

Likewise, if z = f(x, y) = $10x^2y - 12x + 15y^3 + 3yx^3$

$\frac{\partial z}{\partial y} = 10x^2 - 0 + 45y^2 + 3x^2$

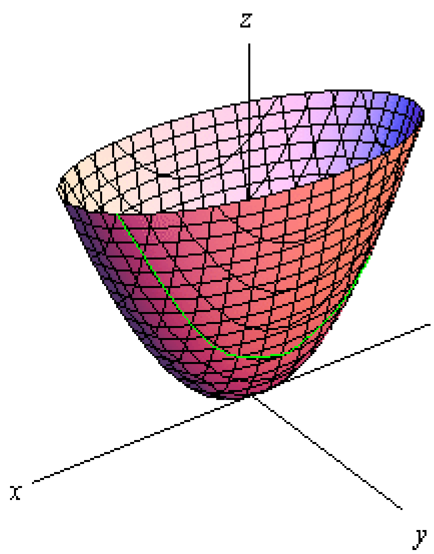
To see what this idea means geometrically, let's return to our friend the elliptic paraboloid and assume an equation of:

$$\frac{x^2}{9} + \frac{y^2}{4} = z$$



If we take y to be a constant value of 2, then we have $\frac{x^2}{4} + 1 = z$

This is the trace shown in the green line below:

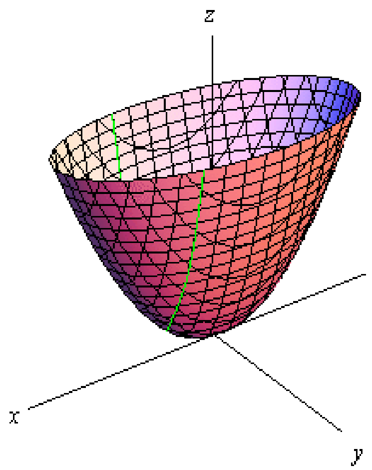


What if we want the slope of this green line at some coordinates (x,y)? All we need to do is take the partial derivative of the original function, $\frac{x^2}{9} + \frac{y^2}{4} = z$

$\frac{\partial z}{\partial x} = \frac{2}{9} \cdot x$ and this agrees with our visual inspection of the curve:

When x = 0, we see the slope of the green line is zero. When x is negative, the slope is negative and when y is positive, the slope is positive. And when x decreases in magnitude, the slope also decreases in magnitude.

Naturally, we could repeat the process for some trace where x is a constant:



$\frac{\partial z}{\partial y} = \frac{2}{4} \cdot y$ where the higher coefficient agrees with a more rapidly changing slope

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Suppose we want $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function $3x^2 + 5y^2 + z^3 + 10xyz = 4$. We just use implicit differentiation. For $\frac{\partial z}{\partial x}$:

$\frac{\partial}{\partial x} 3x^2 = 6x$

$\frac{\partial}{\partial x} 5y^2 = 0$ because we treat *y* as a constant

$\frac{\partial}{\partial x} z^3 = \frac{\partial z}{\partial x} \cdot \frac{\partial}{\partial x} z^3 = \frac{\partial z}{\partial x} \cdot 3z^2$

$\frac{\partial}{\partial x} 10xyz = (10y) \cdot [\frac{\partial}{\partial x} (x)(z)] = 10xy \cdot \frac{\partial z}{\partial x} + 10yz$ by the product rule

$\frac{\partial}{\partial x} 4 = 0$

So $6x + \frac{\partial z}{\partial x} \cdot 3z^2 + 10xy \cdot \frac{\partial z}{\partial x} + 10yz = 0$ or $\quad \frac{\partial z}{\partial x} = -\frac{6x+10yz}{3z^2+10xy}$

Likewise, you will find $\frac{\partial z}{\partial y} = -\frac{10y+10xz}{3z^2+10xy}$

Of course, we can extend these ideas to functions of more than two variables. For instance, suppose $h = 10x^2 + 5xy – 20yz^3$, then

$$\frac{\partial h}{\partial x} = 20x + 5y$$

We can also take higher derivatives. Suppose we have the function $f(x,y) = x^4 + 2x^2y^3 + 4y$

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy^3$$

This can also be written: $f_x = 4x^3 + 4xy^3$, the subscript indicating differentiation with respect to x.

Also, $\frac{\partial f}{\partial y} = 6x^2y^2 + 4$ or $f_y = 6x^2y^2 + 4$

Then $\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\ x^4 + 2x^2y^3 + 4y = \frac{\partial}{\partial x}(4x^3 + 4xy^3) = 12x^2 + 4y^3$ or $f_{xx} = 12x^2 + 4y^3$

$\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\ x^4 + 2x^2y^3 + 4y = \frac{\partial}{\partial y}(6x^2y^2 + 4) = 12x^2y$ or $f_{yy} = 12x^2y$

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Notice that, using the second type of notation, the first subscript is the first derivative taken

$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\ x^4 + 2x^2y^3 + 4y = \frac{\partial}{\partial y}(4x^3 + 4xy^3) = 12xy^2$ or $f_{xy} = 12xy^2$

If the function is continuous over a given range, you will find $f_{xy} = f_{yx}$. This is known as Clairaut's theorem.

Even higher order derivatives can be taken, like $\frac{\partial}{\partial y}(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}))$ or f_{xxy}, and you will again find symmetries like $f_{xxy} = f_{xyx}$.

A number of partial differential equations are very useful in scientific contexts. A few are:

Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for harmonic functions

The wave equation: $\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 x}{\partial t^2}$

Laplace's equation in three-dimensions: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$