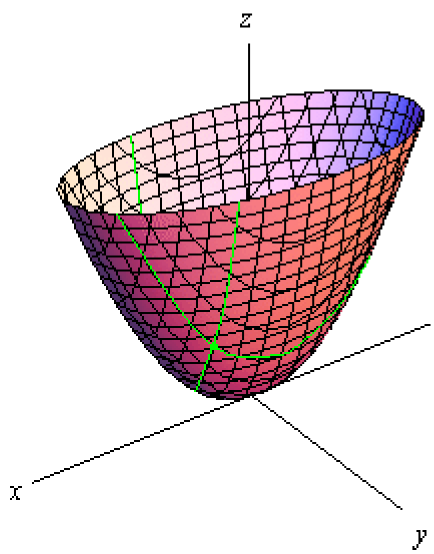
Tangent Planes and Linear Approximations

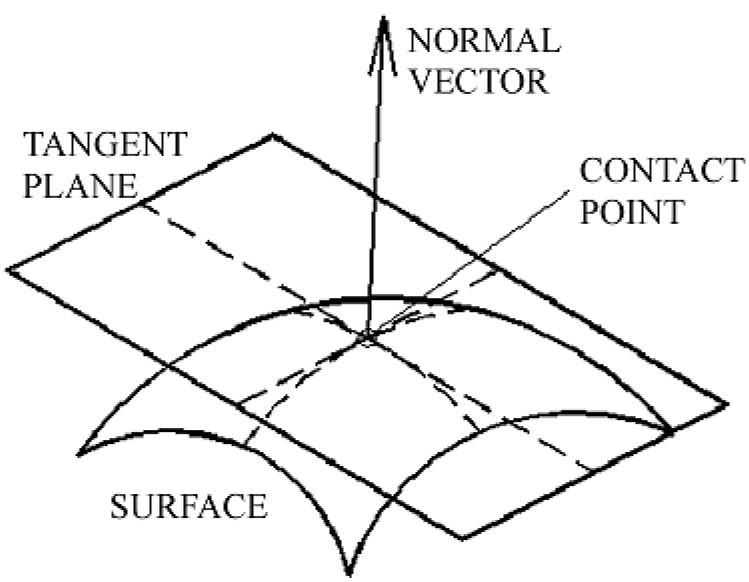
Take the two traces we used for the elliptic paraboloid in the last notes and notice the point where they cross:



The trace along the y-axis meets this point at a tangent line parallel to the y-z plane. The trace along the x-axis meets this point at a tangent line parallel to the x-z plane.

The plane that contains these two tangent lines is the planar approximation of the curve at the point where the traces intersect.

Here is another diagram to represent the idea:



We saw in the notes on planes that one way of defining a plane is to use the scalar equation:

$A\cdot(x - x_0) + B\cdot(y - y_0) + C\cdot(z - z_0) = 0$

which we can rearrange and drop one coefficient if we define $a = -\frac{A}{C}$ and $b = -\frac{B}{C}$ so that:

$z - z_0 = a(x - x_0) + b(y - y_0)$

This tangent plane intersects the plane of $y = y_0$ (or $y - y_0 = 0$), and this intersection of two planes is the tangent line parallel to the y-z plane, thus having the equation:

$z - z_0 = a(x - x_0)$

This line has a slope of *a* which is, conceptually, a change along the z-axis divided by a change along the x-axis, holding the y-value constant. And we have defined this as $\frac{\partial z}{\partial x}$.

Likewise, the tangent intersects the plane of $x = x_0$ (or $x - x_0 = 0$), and this intersection of two planes is the tangent line parallel to the x-z plane, thus having the equation:

$z - z_0 = b(y - y_0)$

This line a slope of *b* which is, conceptually, a change along the z-axis divided by a change along the y-axis, holding the x-value constant. And we have defined this as $\frac{\partial z}{\partial y}$.

The important, end result is this:

The equation for a tangent plane of a surface, $z = f(x,y)$ at a point, $P = (x_0, y_0, z_0)$ is:

$$z - z_0 = \frac{\partial z}{\partial x}\cdot(x - x_0) + \frac{\partial z}{\partial y}\cdot(y - y_0)$$

For an example, let's use our elliptic paraboloid of $\frac{x^2}{9} + \frac{y^2}{4} = z$.

What is the equation of the tangent plane at point (3, 2, 2)?

$\frac{\partial z}{\partial x} = \frac{2}{9}x$ at $x = 3$, $y = 2$ $\frac{\partial z}{\partial x} = \frac{2}{3}$

$\frac{\partial z}{\partial y} = \frac{1}{2}y$ at $x = 3$, $y = 2$ $\frac{\partial z}{\partial y} = 1$

$z - z_0 = \frac{\partial z}{\partial x}\cdot(x - x_0) + \frac{\partial z}{\partial y}\cdot(y - y_0)$ becomes $z - 2 = \frac{2}{3}(x - 3) + 1(y - 2)$ or

$z = \frac{2}{3}x + y - 2$ which is also called the *linearization of the function, L(x,y).*

As mentioned, the tangent plane is an approximation of the curve at the point where the two are in contact, thus it is called the *linear approximation* or *tangent plane approximation* of the curve *f* at point *P*.

To demonstrate this point, let's compare our two functions:

Elliptic paraboloid: $\frac{x^2}{9} + \frac{y^2}{4} = z$

Tangent plane approximation: $z = \frac{2}{3}x + y - 2$

At the point (3, 2), they produce the same value for z, as we expect.

At the point (3.1, 2.1), the curve equation produces z = 2.17, the plane equation produces $2.1\overline{6}$, so a fairly good approximation.

At the point (4, 3), the curve equation produces z = 4.03, the plane equation produces 3.67, a fairly poor approximation.

It is, of course, important that the curve be continuous at the point in question so that the functions are differentiable.

In single-variable calculus, we had $\frac{dy}{dx} = f''(x)$ or $dy = f'(x)\cdot dx$

In multivariable calculus, we simply extend this to:

$dz = \frac{\partial z}{\partial x}(dx) + \frac{\partial z}{\partial y}(dy)$

or, for any number of variables:

$dh = \frac{\partial h}{\partial x}(dx) + \frac{\partial h}{\partial y}(dy) + \frac{\partial h}{\partial z}(dz) + \ldots$ where dh is known as the *differential* of h.

As an example, suppose a cylindrical can has a height of 10cm and a radius of 5cm. Estimate the uncertainty in volume if each measurement has an uncertainty of ±0.10cm.

$V = (h)(\pi\cdot R^2)$

$dV = \frac{\partial V}{\partial h}(dh) + \frac{\partial V}{\partial R}(dR)$

$dV = (\pi R^2)(dh) + (2h\pi R)(dR)$

$dV = (\pi\cdot 5^2)(0.20) + (2\cdot 10\cdot \pi\cdot 5)(0.20) \approx 78.5\text{cm}^3$