Chain Rule

In single-variable calculus, you have the chain rule as:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

where x is a function of t and y is a function of x.

In multivariable calculus, we simply extend this, for z = f(x,y), to:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

where x and y are both functions of t. If z was a function of more variables, we would simply add respective terms to the equation above.

For example, if we have:
$z = x^2 + y^2 + xy$
$x = \sin(t)$
$y = e^t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (2x + y) \cdot \cos(t) + (2y + x) \cdot e^t$$

Suppose we now have a function z that depends upon functions x and y: z = f(x,y)
Furthermore, both functions x and y depend upon s and t: x = f(s,t) and y = f(s,t).

If we want $\frac{\partial z}{\partial t}$, we do so by holding s constant and the chain rule becomes:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

If we want $\frac{\partial z}{\partial s}$, we do so by holding t constant and the chain rule becomes:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Here, z is the dependent variable, s and t are the independent variables, and x and y are known as intermediate variables. Again, if z was a function of more variables, we would add respective terms to the equations above.

For an example, let's take the following functions:
$z = x^2y^3$
$x = s\cdot\cos(t)$
$y = s\cdot\sin(t)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (2xy^3)(-s\cdot\sin(t)) + (3x^2y^2)(s\cdot\cos(t))$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (2xy^3)(\cos(t)) + (3x^2y^2)(\sin(t))$$

Now suppose we have some function that depends upon x and y such that F(x,y) = 0. We also stipulate that y is a function of x: y = *f*(x), so that F(x, f(x)) = 0.

By the chain rule, we can differentiate both sides to produce:

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad \text{where } \frac{dx}{dx} = 1$$

Rearranging, we have:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

As an example, let's find $\frac{dy}{dx}$ for the function $y\cdot\cos(x) = x^2 + y^2$.

$F = x^2 + y^2 - y\cdot\cos(x) = 0$

$\frac{\partial F}{\partial x} = 2x + y\cdot\sin(x)$

$\frac{\partial F}{\partial y} = 2y - \cos(x)$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x + y\cdot\sin(x)}{2y - \cos(x)}$$

As we have done before, we can extend this to more variables, so if y is a function of x and z is a function of both x and y: z = *f*(x,y).

If F(x,y,z) = 0 then

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

where $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$ because we hold y constant when solving for $\frac{\partial z}{\partial x}$

Therefore, $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$

Likewise, $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

For an example, we'll take the function $e^z = xyz$ and find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$F = xyz - e^z = 0$

$\frac{\partial F}{\partial x} = yz$

$\frac{\partial F}{\partial y} = xz$

$\frac{\partial F}{\partial z} = xy - e^z$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{yz}{xy - e^z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{xz}{xy - e^z}$$