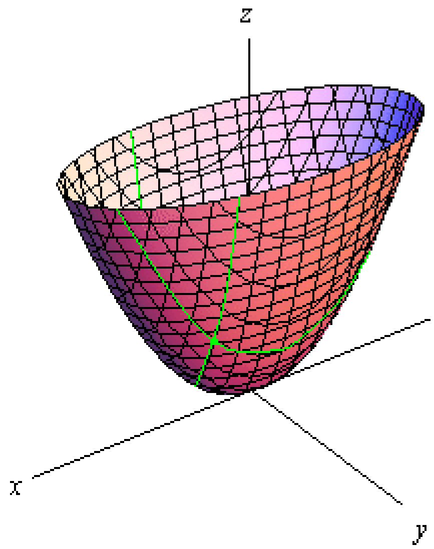
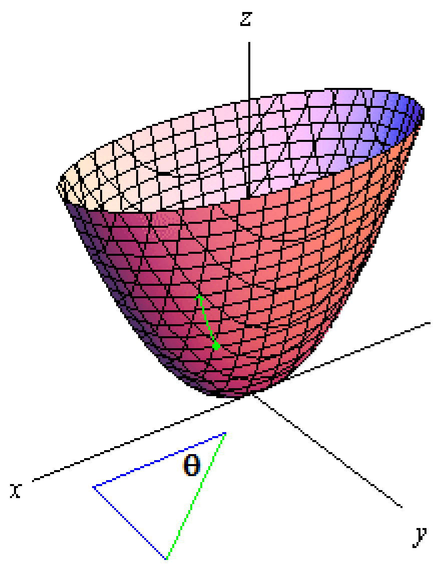
Directional Derivatives and Gradient Vectors

In the notes on tangent planes, we found how to find the slope of the tangent line to some curve if that tangent line is parallel to the yz-plane or the xz-plane.

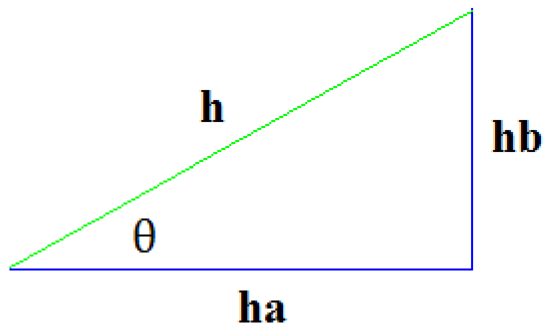


We do so by taking the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

But what if we want the slope of the tangent line to some curve that is not parallel to either plane, but rather some curve that casts an angle θ upon the xy plane?



If we redraw that triangle as seen from above, we can label the sides as such:



where the unit vector **u** = < a, b >

Then $x - x_0 = ha$ or $x = x_0 + ha$
$y - y_0 = hb$ $y = y_0 + hb$

If the height¸ z, above the xy-plane is a function of both x and y, then:

$$\frac{\Delta z}{h} = \frac{z_0 - z}{h} = \frac{f(x,y) - f(x_0,y_0)}{h} = \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

And we define the directional derivative, $D_u f(x,y)$ as $\lim_{h \to 0} \frac{f(x_0+ha,\ y_0+hb) - f(x_0,y_0)}{h}$

Now, as an intermediate, let's define $g(h) = f(x_0 + ha, y_0 + hb)$ so that the derivative at zero:

$g'(0) = \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$ which we have just defined to be $D_u f(x_0,y_0)$

But, by the chain rule, $g'(h) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh} = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b$ or sometimes written $f_x \cdot a + f_y \cdot b$

And at the point where h=0, $x = x_0$ and $y=y_0$, so $g'(0) = f_x(x_0,y_0) \cdot a + f_y(x_0,y_0) \cdot b$

Altogether, $g'(0) = f_x(x_0,y_0) \cdot a + f_y(x_0,y_0) \cdot b = \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = D_u f(x,y)$, so:

$D_u = f_x(x,y) \cdot a + f_y(x,y) \cdot b$

Again, D_u is this slope of the tangent line we're interested in and is called the *directional derivative*.

If the angle is known, it follows from the unit vector mentioned above, that:

$D_u = f_x(x,y) \cdot \cos\theta + f_y(x,y) \cdot \sin\theta$

For example, suppose we have the function $f(x,y) = x^3 - 3xy + 4y^2$ which creates a curved surface in space. What is the slope of the tangent line at point (1,2) which is pointing in the direction of $\frac{\pi}{6}$?

$D_u = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta = (3x^2 - 3y + 0)\cdot\cos(\frac{\pi}{6}) + (0 - 3x + 8y)\cdot\sin(\frac{\pi}{6}) =$

$(3\cdot 1^2 - 3\cdot 2 + 0)(\frac{\sqrt{3}}{2}) + (0 - 3\cdot 1 + 8\cdot 2^2)(\frac{1}{2}) = \frac{13-3\sqrt{3}}{2}$

Or suppose we have the function $f(x,y) = x^4 - x^2\cdot y^3$. What is the slope of the tangent line at point (2,1) along the vector v = **i** + 3**j**?

If we want to use $D_u = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b$, we need to first find a unit vector that has the same direction as the vector v = **i** + 3**j**. Visualizing a right triangle with sides of 1 and 3, the hypotenuse must be $\sqrt{10}$. To force this into a unit hypotenuse, we divide all sides by $\sqrt{10}$ to find that $a = \frac{1}{\sqrt{10}}$ and $b = \frac{3}{\sqrt{10}}$.

Then $\frac{\partial z}{\partial x} = 4x^3 - 2x\cdot y^3$ and $\frac{\partial z}{\partial y} = -3x^2y^2$. Inserting values for a, b, x, and y, we get

$D_u = \frac{-8}{\sqrt{10}}$

We could have also first found the angle and then used sin and cos.

Because both *a* and $\frac{\partial f}{\partial x}$ are along the x-axis while *b* and $\frac{\partial f}{\partial y}$ are along the y-axis, we can write as a dot product: $D_u = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b = \langle \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \rangle \bullet \langle a,b \rangle = \langle \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \rangle \bullet \mathbf{u}$

The first vector of this dot product, $\langle \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \rangle$ is called the *gradient* of f and is written as ∇f which is read as "del f".

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\cdot\mathbf{i} + \frac{\partial f}{\partial y}\cdot\mathbf{j}$$

where, again,

$D_u f(x,y) = \nabla f(x,y)\bullet\mathbf{u}$

If you imagine yourself standing on a curved hill, the gradient tells you how much the altitude is changing if you walk north/south or east/west, but you need to choose a particular direction of your step (which is what the unit vector, **u**, represents) before you can tell how steep the climb is, which is the slope of the tangent line, or the directional derivative.

And, although difficult to visualize, these ideas can be extended so that:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\cdot\mathbf{i} + \frac{\partial f}{\partial y}\cdot\mathbf{j} + \frac{\partial f}{\partial z}\cdot\mathbf{k}$$

where

$D_u f(x,y,z) = \nabla f(x,y,z)\bullet\mathbf{u}$

For a numeric example, suppose the temperature in a region of space follows the function:

$T = 3x^2 + 4y^3 + z$

(a) In what direction does the temperature increase fastest at point (1, 2, 3)?
(b) What is the maximum rate of increase?

(a) First, find the gradient $\nabla T = \frac{\partial T}{\partial x}\cdot\mathbf{i} + \frac{\partial T}{\partial y}\cdot\mathbf{j} + \frac{\partial T}{\partial z}\cdot\mathbf{k} = 6x\cdot\mathbf{i} + 12y^2\cdot\mathbf{j} + \mathbf{k}$

At (1, 2, 3), this will be: 6**i** + 48**j** + 3**k**.

Because this vector is the direction of the gradient, it is, as mentioned above, the same as the direction of the greatest rate of change. We could write it as a unit vector:

$$\frac{2\mathbf{i} + 16\mathbf{j} + \mathbf{k}}{3\sqrt{29}}$$

(b) The rate of change is the length of the gradient vector, 6**i** + 48**j** + 3**k**, which is $9\sqrt{29}$.
We can specify a surface in space with some function F(x,y,z) = k and then choose a point $P = (x_0, y_0, z_0)$ on that surface. If we let a curve *C* pass through that point, we can define the curve with parametric equations **r**(t) = (**x**(t), **y**(t), **z**(t)) so that the position at t=0 is point P.

If the curve lies on the surface, then any point on the curve must also satisfy points of the surface:

F(x(t), y(t), z(t)) = k

Differentiating both sides with the chain rule produces:

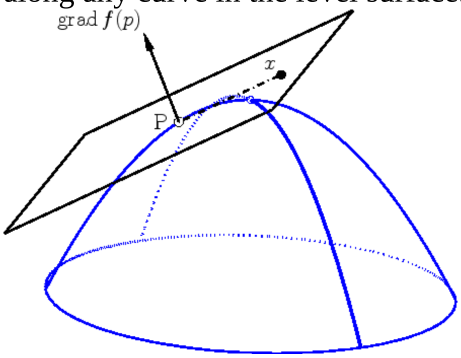
$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0$$

And if $\nabla F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$ while $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$

Then the dot product $\nabla F \bullet \frac{d\boldsymbol{r}}{dt} = 0$

And, specifically, at point P, $\nabla F(x_0, y_0, z_0) \bullet \mathbf{r}'(t_0) = 0$

Remember that the dot product between two vectors is zero when they are perpendicular, which is what the equation above informs us: the gradient vector is perpendicular to the tangent vector along any curve in the level surface.



We can thus define the tangent plane to the level surface as the plane that passes through P and has the normal vector of the gradient, $\nabla F(x_0, y_0, z_0)$. All positions in this plane will then have coordinates (x, y, z) such that:

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(where F_x is another notation for $\frac{\partial F}{\partial x}$)

Example: An ellipsoid has the equation $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$. For the point on the surface of the ellipsoid, (-2, 1, -3), find the plane that passes through that point and the line normal to that plane.

First, find the gradient, $\nabla F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = \frac{x}{2} + 2y + \frac{2z}{9}$

Then $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$ becomes

$\frac{-2}{2} \cdot (x + 2) + 2 \cdot (y - 1) + \frac{-2}{3} \cdot (z + 3) = 0$ or $x - 2y + \frac{2}{3}z + 6 = 0$

We can write the normal line as a series of parametric equations:

$x = -2 + 3t$ $y = 1 - 6t$ $z = 3 + \frac{2}{3}t$

As an application, consider the very useful equation in electrostatics: E = - $\frac{dV}{dR}$. In gradient notation, this would be:

E = -∇(V)

A lone charge produces an electric field in its vicinity according to the equation:

$$E = \frac{k \cdot Q}{R^2} \cdot \hat{R}.$$

This tells us that the electric field is constant at any point on a sphere that surrounds the charge concentrically and points outward away from the charge.

Electric potential follows the equation:

$$V = \frac{k \cdot Q}{R}$$

So if you walk around on one of these spherical surfaces, your electric potential is constant. But if you move directly towards the charge, you are moving in the direction of the gradient vector, because that is the direction in which the electric field increases most rapidly.