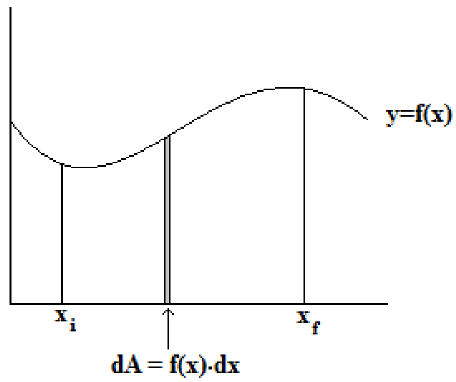
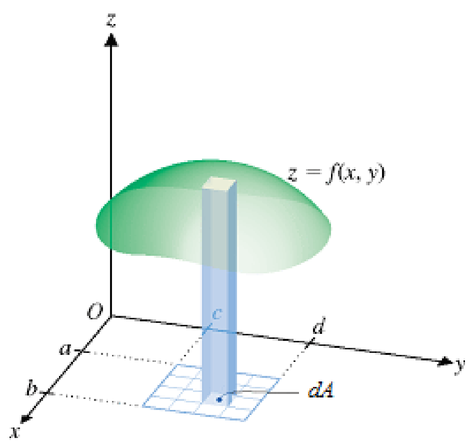
Introduction to double integrals and iterated integrals



In single-variable calculus, you find the area under a curve by imagining the area as an infinite number of vertical strips and then finding the sum of the areas of these strips:

dA = f(x)·dx so $A = \int_{x_i}^{x_f} f(x) \cdot dx$

With two variables, the process is analogous. Imagine a surface over the x-y plane with a height, *z*, that is a function of both *x* and *y*, z = f(x,y).



The volume of this column would then be:
dV = f(x,y)·dA

and the total volume between the boundaries a, b, c, and d would be:

$V = \int_{y_i}^{y_f} \int_{x_i}^{x_f} f(x,y) \cdot dA$

which is the same as

$V = \int_{y_i}^{y_f} \int_{x_i}^{x_f} f(x,y) \cdot dx \cdot dy$

The above is known as an *iterated integral* and can be solved fairly simply if you think of it as:

V = $\int_{y_i}^{y_f}[\int_{x_i}^{x_f} f(x,y) \cdot dx\ ]\ dy$

First, integrate the internal bracketed term with respect to x, holding the y-value constant. This is just the opposite of taking the partial derivative of a function with respect to x, holding the y-value constant. Once the bracket has been integrated, simply integrate with respect to y.

For example, find the value of:

V = $\int_0^3 \int_1^2 (x^2y) \cdot dx \cdot dy$

Evaluating the inner integral with respect to x produces:

V = $\int_0^3 \int_1^2 (x^2y) \cdot dx \cdot dy$ = $\int_0^3 [\frac{1}{3} \cdot 2^3 \cdot y - \frac{1}{3} \cdot 1^3 \cdot y] \cdot dy$ = $\int_0^3 [\frac{7}{3} \cdot y] \cdot dy$

Then, evaluating the outer integral with respect to y, we have:

V = $\frac{7}{6} \cdot 3^2$ - $\frac{7}{6} \cdot 0^2$ = $\frac{21}{2}$

Of course, this volume is independent of which way we integrated, so we would get the same result by integrating with respect to y as the inner-integral and with respect to x as the outer-integral.

We could have also factored the function x^2y as $(x^2)(y)$ so that:

V = $\int_0^3 \int_1^2 (x^2y) \cdot dx \cdot dy$ = $\int_0^3 y \cdot \int_1^2 x^2 \cdot dx \cdot dy$ = $\int_0^3 y \cdot dy \int_1^2 x^2 \cdot dx$ = $\frac{21}{2}$

And if we want the average height of the surface, we only need to divide the volume by the surface area in the x-y plane, which here is (3-0)(2-1) = 3.

h_{avg} = $\frac{21}{6}$ = $\frac{7}{2}$

Lastly, as is true for single-variable calculus, there are a couple useful properties of double integrals:

$\iint [f(x,y) + g(x,y)]dA$ = $\iint f(x,y)dA$ + $\iint g(x,y)dA$

$\iint c \cdot f(x,y)dA$ = c$\cdot \iint f(x,y)dA$