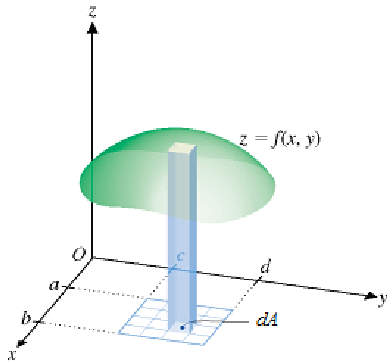
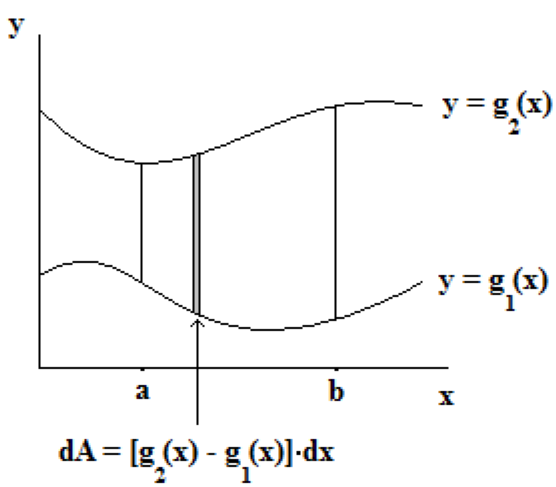
Double integrals

In the previous notes, we saw how to calculate the volume beneath a function f(x,y) above a rectangle in the x-y plane:



We would now like to generalize this process so that we can find similar volumes above more complex shapes in the x-y plane.

Suppose such a shape in the x-y plane is bounded at the top and bottom by the functions $g_2(x)$ and $g_1(x)$ with horizontal boundaries a and b:

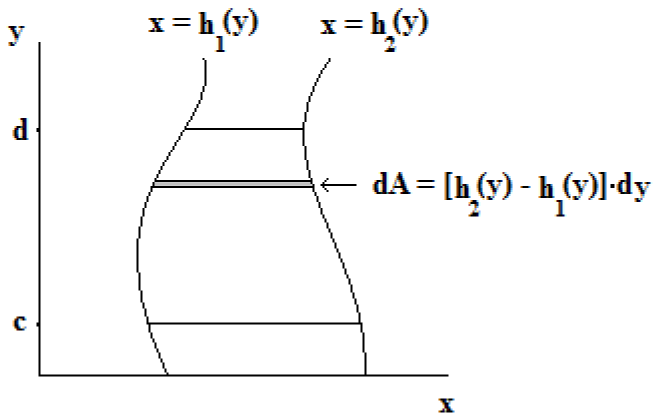


We can see such an enclosed volume would be written as:

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \cdot dy \cdot dx$$

Notice that the inner integral is assuming a fixed value for x as we integrate the strip along the y-dimension.

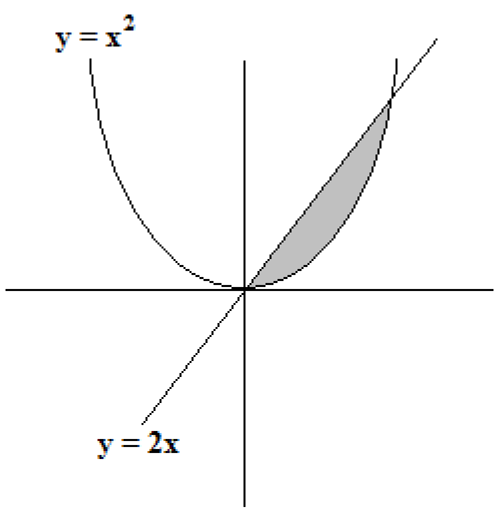
Likewise, if we had some enclosed space in the x-y plane defined by $h_2(y)$ and $h_1(y)$:



$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \cdot dx \cdot dy$$

For example, let's find the volume of the solid that lies under the surface of $z = x^2 + y^2$ and is above a region in the x-y plane bounded by $y = 2x$ and $y = x^2$.

First, it's a good idea to draw the bounded shape in the x-y plane to assist our translating the problem into a double integral:



We can then calculate the left and right boundaries of this shaded area with $x^2 = 2x$ so that $x_1 = 0$ and $x_2 = 2$.

We have a choice of integrating vertical strips from left to right or integrating horizontal strips from bottom to top. Following the more common process, I will use the first:

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \cdot dy \cdot dx$$

$$V = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \cdot dy \cdot dx$$

The inner integral will become:

$[x^2(2x) + \frac{1}{3}(2x)^3] – [x^2(x^2) + \frac{1}{3}(x^2)^3] = -\frac{1}{3}x^6 – x^4 + \frac{14}{3}x^3$

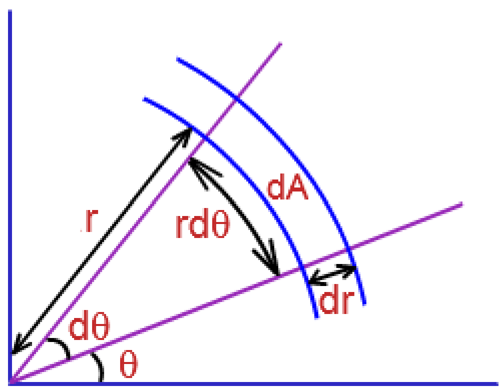
Which makes the outer integral:

$V = \int_0^2 (-\frac{1}{3}x^6 - x^4 + \frac{14}{3}x^3) \cdot dx = -\frac{128}{21} - \frac{32}{5} + \frac{224}{12} \approx 6.17$

It's also possible to find enclosed volumes in a very similar manner using polar coordinates. Recall from the second set of notes, we had:

$R^2 = x^2 + y^2$
$x = R \cdot \cos\theta$
$y = R \cdot \sin\theta$

We can then see that, instead of $dA = dy \cdot dx$, which was true for Cartesian coordinates, we have:
$dA = r \cdot dr \cdot d\theta$



If these small patches of defined area exist in the x-y plane with some surface function f(x,y) above the x-y plane, then the volume enclosed by the surface function and defined area would be:

$$V = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(x,y) \cdot r \cdot dr \cdot d\theta$$

or

$$V = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r \cdot cos\theta, r \cdot sin\theta) \cdot r \cdot dr \cdot d\theta$$

For example, let's find the volume below the paraboloid $z = 1 - x^2 - y^2$ and above the x-y plane.

First note that when we are in the x-y plane, z = 0 and the equation becomes the equation of a circle, $x^2 + y^2 = 1$. The integral then becomes:

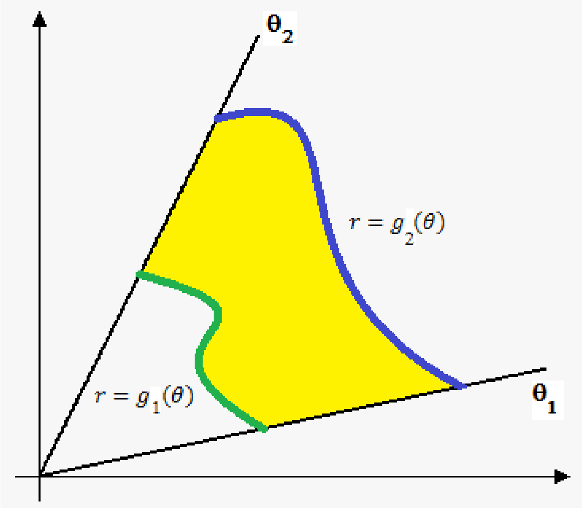
$$V = \int_0^{2\pi} \int_0^1 (1 - r^2) \cdot r \cdot dr \cdot d\theta$$

Remembering that the parenthetic term is z = f(x,y), which here is $z = 1 - x^2 - y^2 = 1 - r^2$

The inner integral is then $\frac{1}{2}(1^2) - \frac{1}{4}(1^4) = \frac{1}{4}$

The outer integral is then $\frac{1}{4}(2\pi) = \frac{\pi}{2}$

Lastly, in the same way we extended double integrals from a rectangular base to an irregular base in Cartesian coordinates, we can something similar with polar coordinates:



$$V = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(x,y) \cdot r \cdot dr \cdot d\theta$$