Applications of double integrals, surface area

Suppose we have a flat disk of some shape and the density of the disk is a function of the individual positions of the disk, that is, $\sigma = \frac{dm}{dA} = f(x,y)$. We can find the total mass of the disk by multiplying the density of each infinitely small piece of area times the density at that particular location, $dm = \sigma \cdot dA$.

This makes the total mass of the disk:

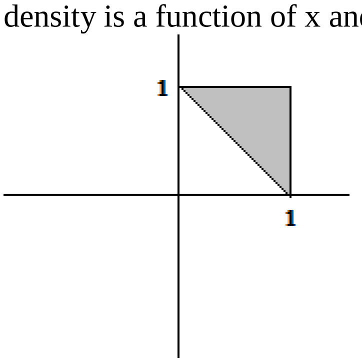
$m = \iint \sigma(x,y) \cdot dA$ or $m = \iint \sigma(x,y) \cdot dx \cdot dy$

This is very similar to what we have done before, finding volumes in space of shapes where the height of the shape along the z-axis depends upon the position in the x-y plane. Here, we are simply replacing the concept of height with the concept of density.

Likewise in electrostatics, if we define charge density as charge per area: $\sigma = \frac{dQ}{dA} = f(x,y)$, then the total charge of the system is similarly:

$Q = \iint \sigma(x,y) \cdot dA$ or $Q = \iint \sigma(x,y) \cdot dx \cdot dy$

For example, suppose we have a triangle with vertices of (0,1), (1,1) and (1,0) where the charge density is a function of x and y such that $\sigma(x,y) = xy$. What is the total charge?



$$Q = \int_0^1 \int_{1-x}^1 (x \cdot y) \cdot dy \cdot dx$$

Note the boundaries of the inner integral arise because we want to use vertical strips of area of the triangle and not the entire square. The outer integral is then the sum of all of these vertical strips.

The inner integral produces $\frac{x}{2} - \frac{x}{2}(1-x)^2 = x^2 - \frac{x^3}{2}$ which makes the outer integral $\frac{x^3}{3} - \frac{x^4}{8}$ with boundaries of 0 and 1, producing a total charge of $\frac{5}{24}$.

A third application in physics utilizes the concept of *center-of-mass*. In the x-dimension, this is defined at:

$$x_{cm} = \frac{\int x \cdot dm}{\int dm}$$

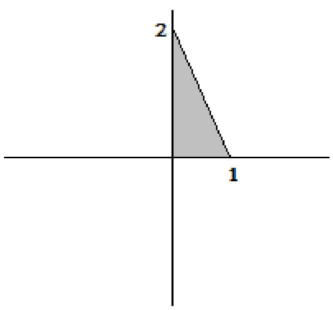
which is, conceptually, the average position of a system, weighted by mass

$$y_{cm} = \frac{\int y \cdot dm}{\int dm}$$

is the similar equation in the y-dimension

If the mass density of the system varies with position, we must then write $dm = \sigma(x,y) \cdot dA$ and use this in our integrals.

For example, let's find the center of mass for a triangle with vertices of (0,0), (0,2), and (1,0) which has a density function of $\sigma = 1 + 3x + y$.



First, let's find the total mass of the triangle which will serve as the denominator for each center of mass coordinate:

$$M = \int dm = \int_0^1 \int_0^{2-2x} (1 + 3x + y) \cdot dy \cdot dx$$

The inner integral is $y + 3xy + \frac{1}{2}y^2$ which has bounds of 0 and 2-2x, so it becomes $4 - 4x^2$. Integrating that with respect to x becomes $4x - \frac{4}{3}x^3$ with bounds of 0 and 1 equals $\frac{8}{3}$.

For the center of mass in the x-dimension, the numerator would be:

$$\int x \cdot dm = \int_0^1 \int_0^{2-2x} x \cdot (1 + 3x + y) \cdot dy \cdot dx$$

The inner integral is $xy + 3x^2y + \frac{1}{2}xy^2$ with bounds of 0 and 2-2x, which becomes $4x - 4x^3$. Integrating that with respect to x becomes $2x^2 - x^4$ with bounds of 0 and 1 equals 1.

With a numerator of 1 and a denominator of $\frac{8}{3}$, the center of mass in the x-dimension is $\frac{3}{8}$. Follow a similar process for the y-dimension and you will find a coordinate of $\frac{11}{16}$.

Another useful equation in physical mechanics is rotational inertia, a measure of how difficult it is to give an object angular acceleration around a defined axis. From the fundamental definition of $I = m \cdot R^2$, we can write:

$dI = R^2 \cdot dm = R^2 \cdot \sigma(x,y) \cdot dA$

If we want the rotational inertia around the x-axis, then R = y and the above equation becomes:

$I_x = \iint y^2 \cdot \sigma(x,y) \cdot dy \cdot dx$

Rotational inertia around the y-axis would be:

$I_y = \iint x^2 \cdot \sigma(x,y) \cdot dy \cdot dx$

And rotational inertia around the origin would be:

$I = \iint (x^2 + y^2) \cdot \sigma(x,y) \cdot dy \cdot dx$

For example, let's find all three rotational inertias for the curve between $y = 1 – x^2$ and y = 0 which has a density function of $\sigma(x,y) = ky$.

$I_x = \iint y^2 \cdot \sigma(x,y) \cdot dy \cdot dx = \int_{-1}^{1} \int_{0}^{1-x^2} y^2 \cdot ky \cdot dy \cdot dx$

The inner integral becomes $\frac{1}{4}ky^4$ with bounds of 0 and $1 – x^2$ which is $\frac{1}{4}k(1 – x^2)^4$.

We expand this and integrate with respect to x between bounds of -1 and 1 to achieve $\frac{64}{315}k$.

$I_y = \iint x^2 \cdot \sigma(x,y) \cdot dy \cdot dx = \int_{-1}^{1} \int_{0}^{1-x^2} x^2 \cdot ky \cdot dy \cdot dx$

The inner integral becomes kx^2y^2 with bounds of 0 and $1 – x^2$ which is $kx^2(1 – x^2)^2$.

Expanding this and integrating with respect to x between -1 and 1 becomes $\frac{8}{105}k$.

Around the origin, $I = \iint (x^2 + y^2) \cdot \sigma(x,y) \cdot dy \cdot dx = \int_{-1}^{1} \int_{0}^{1-x^2} (x^2 + y^2) \cdot ky \cdot dy \cdot dx$

The inner integral becomes $\frac{1}{2}kx^2y^2 + \frac{1}{4}ky^4$ with bounds of 0 and $1 – x^2$ which is $\frac{1}{2}kx^2(1 – x^2)^2 + \frac{1}{4}k(1 – x^2)^4$.

Expanding this and integrating with respect to x between -1 and 1 produces $\frac{88}{315}k$.

We can easily check this because $dI = dm \cdot R^2 = dm \cdot (x^2 + y^2) = dm \cdot x^2 + dm \cdot y^2$

Therefore, $I_o = I_y + I_x$, which in our problem is: $\frac{88}{315}k = \frac{8}{105}k + \frac{64}{315}k$.

Our fifth application of double integrals involves probability functions. This is very useful in quantum mechanics where a certain particle may exist in a region of space, but *where* in that region is probabilistic.

For example, a particle may exist somewhere in the x-y plane, but the likelihood of existing at some position depends upon the x and y coordinates. We can say the probability density, $\sigma = f(x,y)$. But the particle must exist somewhere, so the total probability of existence is one.

Therefore, $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(x,y) \cdot dy \cdot dx = 1$

Let's suppose the particle is bounded somewhere in the region $0 \le x \le 10$ and $0 \le y \le 10$ and the probability density is $C(x + 2y)$. What is the probability that $x \le 7$ and $y \ge 2$?

$$\int_0^{+10} \int_0^{+10} C(x + 2y) \cdot dy \cdot dx = 1$$

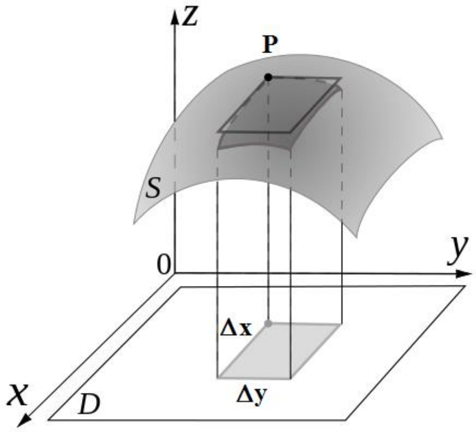
$$\int_0^{+10} (10x + 100) \cdot dx = \frac{1}{C}$$

$500 + 1000 = \frac{1}{C}$ or $C = \frac{1}{1500}$

Then, $\int_0^7 \int_2^{10} \frac{1}{1500} (x + 2y) \cdot dy \cdot dx = P$

$$P = \frac{1}{1500} \int_0^7 (8x + 96) \cdot dx = \frac{868}{1500} \approx 58\%$$

Lastly, let's use double integrals to find the surface area of some surface above the x-y plane.



The area of the parallelogram which approximates the area of the curve underneath, as we saw in the notes on cross-products, is the cross product of the x and y vectors which originate at point P. If we let f_x be the slope of the surface along the x-dimension at P and f_y be the slope of the surface along the y-dimension at P, then:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix} = \Delta x \cdot \Delta y \cdot \mathbf{k} - f_x \cdot \Delta x \cdot \Delta y \cdot \mathbf{i} - f_y \cdot \Delta x \cdot \Delta y \cdot \mathbf{j} = \Delta A(\mathbf{k} - f_x \cdot \mathbf{i} - f_y \cdot \mathbf{k})$$

The magnitude of this quantity is then $\sqrt{f_x^2 + f_y^2 + 1} \cdot \Delta A$

As the parallelogram shrinks to infinitely small, we have:

$$d(\text{Area}) = \sqrt{f_x^2 + f_y^2 + 1} \cdot dA = \sqrt{f_x^2 + f_y^2 + 1} \cdot dy \cdot dx$$

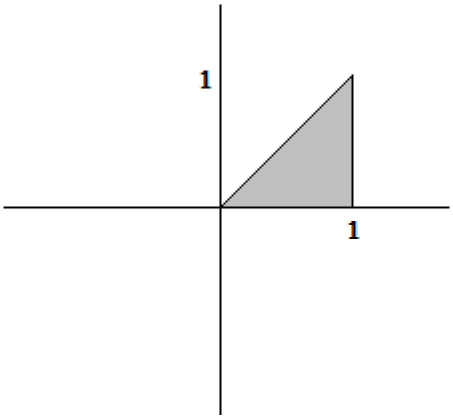
And the area of the surface itself is then:

$$A = \iint \sqrt{f_x^2 + f_y^2 + 1} \cdot dy \cdot dx$$

Written in the form of partial derivatives, this is:

$$A = \iint \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \cdot dy \cdot dx$$

For an example, let's take a triangle in the x-y plane with vertices of (0,0), (1,1) and (1,0). Above this triangle is a surface with a function, $z = x^2 + 2y$. What is the surface area of this surface?



$$A = \iint \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \cdot dy \cdot dx$$

$$A = \int_0^1 \int_0^x \sqrt{(2x)^2 + (2)^2 + 1} \cdot dy \cdot dx$$

$$A = \int_0^1 (x \cdot \sqrt{4x^2 + 5}) \cdot dx$$

$$A = \frac{1}{12}(4 \cdot 1^2 + 5)^{3/2} - \frac{1}{12}(4 \cdot 0^2 + 5)^{3/2} = \frac{27-5\sqrt{5}}{12}$$