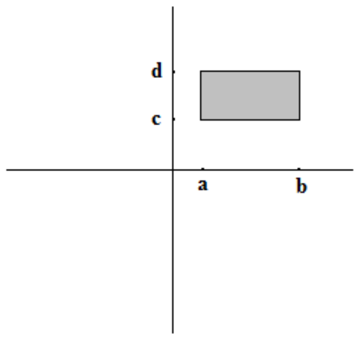
Triple integrals

It is natural to proceed from double integrals, which were often related to surfaces in the x-y plane, to triple integrals, which are related to volumes in space.

For example, we could find the surface area of a rectangle



with A = $\int_a^b \int_c^d dy \cdot dx$

The volume of a rectangular parallelepiped would be similar:

V = $\iiint dz \cdot dy \cdot dx$ given appropriate coordinates of the sides

We can also add a function to the integral as we did with applications of double integrals. For example, suppose the charge density of a region of space follows the function σ(x,y,x) = xyz^2. Find the total charge of a box with sides such that the region of space is contained within:
$0 \leq x \leq 1$
$-1 \leq y \leq 2$
$0 \leq z \leq 3$

Q = $\iiint \sigma(x, y, z)dz \cdot dy \cdot dx$

Q = $\int_0^1 \int_{-1}^2 \int_0^3 (xyz^2) \cdot dz \cdot dy \cdot dx$

Q = $\int_0^1 \int_{-1}^2 (9xy) \cdot dy \cdot dx$

Q = $\int_0^1 \frac{27}{2} x \cdot dx$

Q = $\frac{27}{4}$

You would get the same answer regardless of the order in which you integrated dz, dy, and dx and σ could have just as easily represented mass density, in which case the triple integral would yield the total mass of the box.

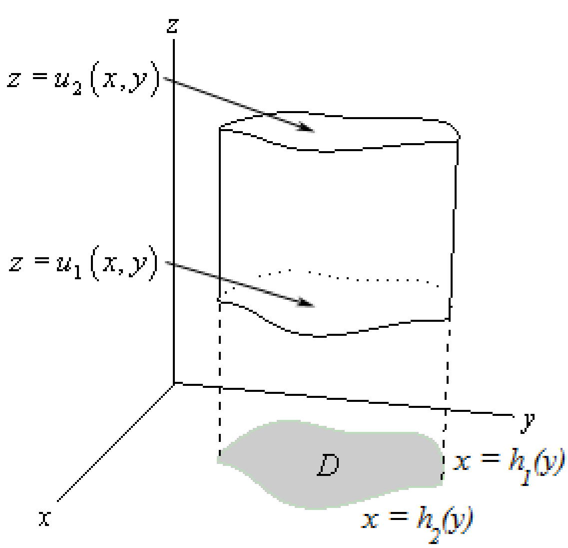
Other applications are easily extended into three dimensions:

	Double integral	Triple integral
Center of mass, x-coordinate	$x_{cm} = \frac{\iint x \cdot \sigma(x,y) \cdot dy \cdot dx}{\iint \sigma(x,y) \cdot dy \cdot dx}$	$x_{cm} = \frac{\iiint x \cdot \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}{\iiint \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}$
Center of mass, y-coordinate	$y_{cm} = \frac{\iint y \cdot \sigma(x,y) \cdot dy \cdot dx}{\iint \sigma(x,y) \cdot dy \cdot dx}$	$y_{cm} = \frac{\iiint y \cdot \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}{\iiint \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}$
Center of mass, z-coordinate		$z_{cm} = \frac{\iiint z \cdot \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}{\iiint \sigma(x,y,z) \cdot dz \cdot dy \cdot dx}$
Rotational inertia around the x-axis	$I_x = \iint y^2 \cdot \sigma(x,y) \cdot dy \cdot dx$	$I_x = \iiint (y^2 + z^2) \cdot \sigma(x,y) \cdot dz \cdot dy \cdot dx$
Rotational inertia around the y-axis	$I_y = \iint x^2 \cdot \sigma(x,y) \cdot dy \cdot dx$	$I_y = \iiint (x^2 + z^2) \cdot \sigma(x,y) \cdot dz \cdot dy \cdot dx$
Rotational inertia around the z-axis		$I_z = \iiint (x^2 + y^2) \cdot \sigma(x,y) \cdot dz \cdot dy \cdot dx$
Rotational inertia around the origin	$I_o = \iint (x^2 + y^2) \cdot \sigma(x,y) \cdot dy \cdot dx$ or $I_o = I_x + I_y$	$I_o = \iiint (x^2 + y^2 + z^2) \cdot \sigma(x,y,z) \cdot dz \cdot dy \cdot dx$ or $I_o = I_x + I_y + I_z$
Probability normalization	$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(x,y) \cdot dy \cdot dx = 1$	$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(x,y,z) \cdot dz \cdot dy \cdot dx = 1$

Of course, the spatial volume in question in unlikely to always have the shape of a rectangular box. If, for instance, the four vertical sides were planar, but the top and bottom were functions of x and y such that $z_{top} = u_2(x,y)$ and $z_{bottom} = u_1(x,y)$, we could find the mass of the shape with:

$$M = \iiint_{u_1(x,y)}^{u_2(x,y)} \sigma(x,y,z) \cdot dz \cdot dy \cdot dx$$

A further complexity would be if two of the four vertical walls were functions themselves:



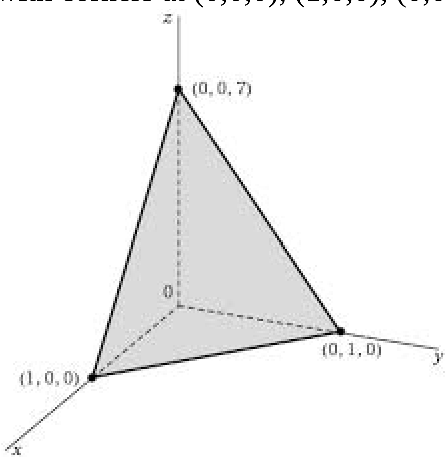
Here the top and bottom, front and back surfaces are all functions.

The integral for mass then becomes:

$$M = \iint_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} \sigma(x, y, z) \cdot dz \cdot dx \cdot dy$$

Consider the order of integration from inside-out. The innermost integral is summing little bits of mass from the bottom surface to the top surface along an infinitely thin column with sides dy and dx. The next integral out is summing these columns in sheets parallel to the x-z plane as we go from h_1 to h_2. The outmost integral is then summing these sheets as you travel parallel to the y-axis from the left-most coordinate to the right-most coordinate.

For practice, we'll try one without the mass density. Let's just find the volume of a tetrahedron with corners at (0,0,0), (1,0,0), (0,0,7) and (0,1,0):



Most of the work is going to be in figuring-out the boundaries of integration. Once we've done that, the remainder is relatively straightforward integration. Conceptually, we're going to first find the volumes of vertical columns with sides of dx and dy. Then we'll sum those columns in sheets parallel to the x-z plane. Lastly, we'll sum those sheets as we travel along the y-dimension. So our integral will look like:

$$V = \int_0^y \int_0^x \int_0^z dz \cdot dx \cdot dy$$

The z-coordinate of the top of our thin columns will be: $z = 7 - x - y$

As we sum these columns parallel to the x-z plane, we are going to travel from the y-axis to $x = 1 - y$

And as we sum these sheets, we are going to travel from the x-axis to $y = 1$. So our integral is:

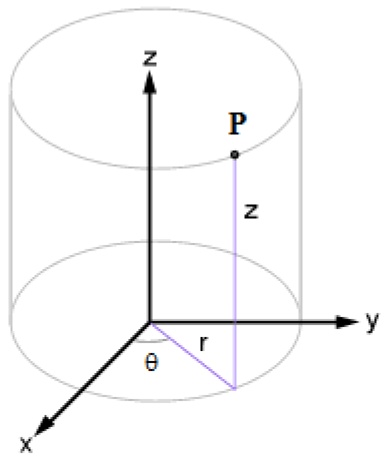
$$V = \int_0^1 \int_0^{1-y} \int_0^{7-x-y} dz \cdot dx \cdot dy$$

The innermost integral is $7 - 7x - 7y$. The next integral out is $7(1 - y) - \frac{7}{2}(1 - y)^2 - 7y(1 - y)$.

The outermost integral is then $\frac{7}{6}$. We can confirm this with another way of finding the volume of a tetrahedron, namely taking the three sides along the coordinates axes as vectors and using:

$$V = \frac{a \bullet (b \times c)}{6}$$

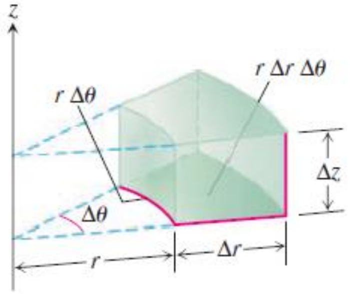
There are other coordinate systems in which we can use triple integrals. One is called *cylindrical coordinates*:



Point P has coordinates (r, θ, z) where r is the distance of the point from the z-axis, θ is the angle in the x-y plane, and z is the height above the x-y plane. All points with a common r create a cylinder.

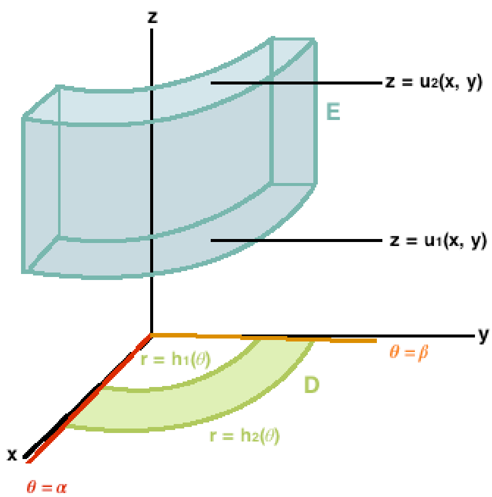
Converting to Cartesian coordinates, you can see:
$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$
$z = z$

From Cartesian coordinates to cylindrical coordinates, you would use:
$r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}(\frac{y}{x})$
$z = z$



The diagram above shows that, in cylindrical coordinates, $dV = r \cdot d\theta \cdot dr \cdot dz$

The volume of a given geometry in cylindrical coordinates, such as:



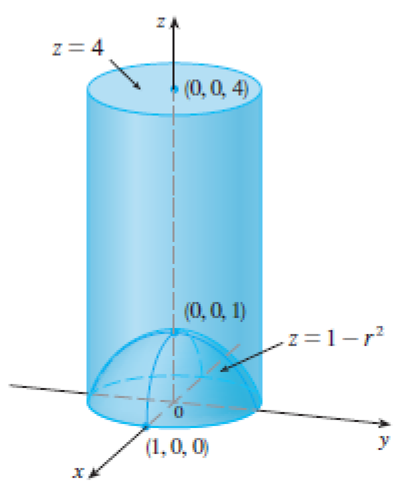
would then be:

$$V = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} dz \cdot r \cdot dr \cdot d\theta$$

And if we know the density of the object as a function of position, ρ = f(r,θ,z), then we just include that in the integral:

$$M = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} \rho(r,\theta,z) dz \cdot r \cdot dr \cdot d\theta$$

For example, let's suppose we have a cylinder of $x^2 + y^2 = 1$ with a flat top at z = 0 and a paraboloid bottom of $z = 1 - x^2 - y^2$ and a density that increases with distance from the z-axis so that ρ = k·r. What is the mass of this cylinder?



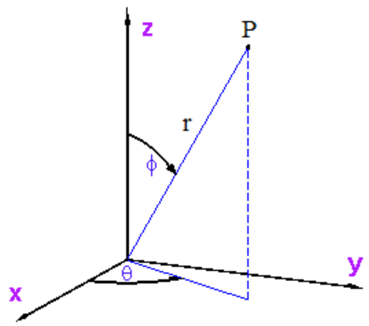
$$M = \int_{0}^{2\pi} \int_{0}^{1} \int_{1-r^2}^{4} (kr) dz \cdot r \cdot dr \cdot d\theta$$

The inner integral gives us krz with bounds of $1 - r^2$ and 2 which is $3kr^2 + kr^4$.

The next integral out gives us $kr^3 + \frac{kr^5}{5}$ with bounds of 0 and 1 which is $\frac{6k}{5}$.

The outermost integral is then $\frac{12\pi k}{5}$.

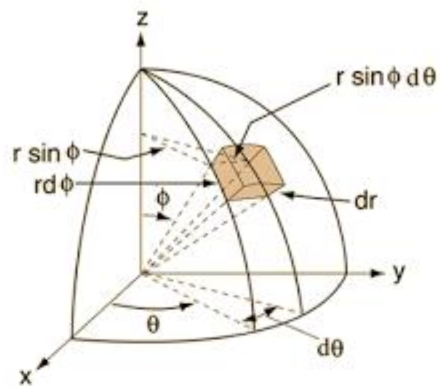
Lastly is a third form of coordinates, *spherical coordinates*:



Point P has coordinates (r, θ, φ) where r is the distance of the point from the origin, θ is the angle in the x-y plane, and φ is the angle above the x-y plane. All points with a common r form a sphere.

Converting to Cartesian coordinates, you would use:
$x = r \cdot \sin\varphi \cdot \cos\theta$
$y = r \cdot \sin\varphi \cdot \sin\theta$
$z = r \cdot \cos\varphi$

Converting from Cartesian to spherical coordinates, you would use:
$r = \sqrt{x^2 + y^2 + z^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$\varphi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$



We can see in the diagram above that a volume element in spherical coordinates is:
$dV = dr \cdot (r \cdot \sin\varphi \cdot d\theta) \cdot (r \cdot d\varphi) = r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

The total volume in spherical coordinates would then be:

$V = \iiint r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

If we had the density of a particular shape as a function of coordinates within that shape, we could insert that into the integral so that:

$M = \iiint \rho(r, \theta, \varphi) \cdot r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

In Cartesian coordinates, this would be:

$M = \iiint \rho(x, y, z) \cdot r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

$M = \iiint \rho(r \cdot \sin\varphi \cdot \cos\theta, r \cdot \sin\varphi \cdot \sin\theta, r \cdot \cos\varphi) \cdot r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

For example, let's find the mass of a sphere with radius 1 that has a density function of $\rho = e^{(x^2+y^2+z^2)^{3/2}}$.

$M = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{(r^2)^{3/2}} \cdot r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{(r^3)} \cdot r^2 \cdot \sin\varphi \cdot dr \cdot d\theta \cdot d\varphi$

When completing the inner integral, sinφ is a constant, so it can be moved to the left of the summa sign and we have:

$M = \int_0^{\pi} \int_0^{2\pi} \sin\varphi \cdot \frac{1}{3} \cdot (e - 1) \cdot d\theta \cdot d\varphi$

In the next inner integral, sinφ is again a constant, so it can be moved left to produce:

$M = \int_0^{\pi} sin\varphi \cdot \frac{1}{3} \cdot 2\pi \cdot (e - 1) \cdot d\varphi$

The final integral then becomes

$M = \frac{4}{3} \cdot \pi \cdot (e - 1)$