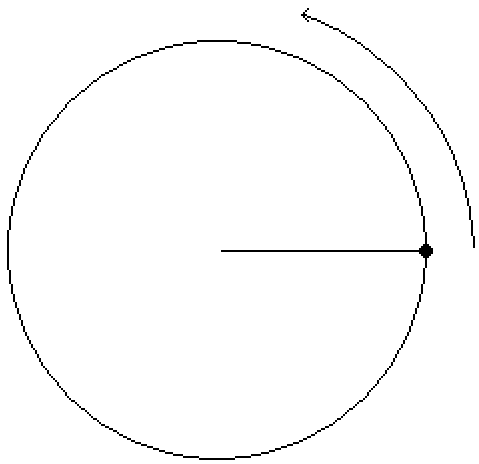
Polar Equations



A stone is tied to a string with length *L* and, as seen from overhead, is swung in a circle with an angular velocity, ω.

If we wanted the position of the stone at any time, *t*, we could use the following Cartesian notation:

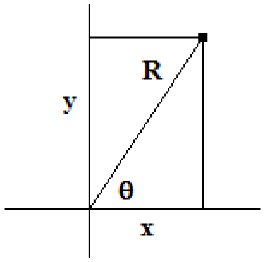
$x = L\cdot\cos(\omega t)$
$y = L\cdot\sin(\omega t)$

or position, $s = [L\cdot\cos(\omega t), L\cdot\sin(\omega t)]$

But a more efficient notation would involve polar coordinates where the first coordinate is the distance from the origin and the second coordinate is the angle from the positive x-axis. The position of the stone in polar coordinates would then simply be:

$s = (L, \omega t)$

You should already be familiar with the steps to translate back and forth from Cartesian to polar coordinates.



Cartesian to polar:

$R^2 = x^2 + y^2$ and $\theta = \tan^{-1}(\frac{y}{x})$

Polar to Cartesian:

$x = R \cdot \cos\theta$
$y = R \cdot \sin\theta$

We will add the following two conventions:

1. In polar coordinates, $(-R, \theta) = (R, \theta+\pi)$. For example, a vector of (-2m, 0 rad) is the same as the vector (2m, π rad).

2. In polar coordinates, $(R, \theta) = (R, \theta + n \cdot 2\pi)$. For example, the vector (5m, 0 rad) is the same as the vector (5m, 2π rad) is the same as the vector (5m, 4π rad), etc.

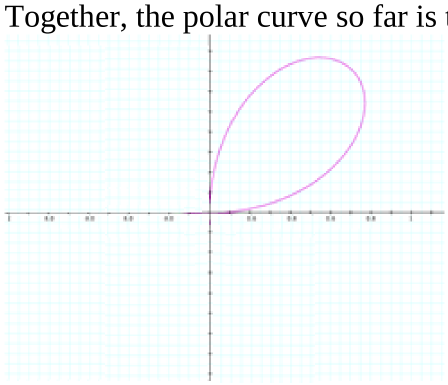
Polar curves take the general form, $R = f(\theta)$, which is to say, the radial distance from the origin is a function of the angle from the positive x-axis.

Let's take a simple example of $R = \sin(2\theta)$.

From $\theta = 0$ to $\theta = \frac{\pi}{4}$ R is positive and increasing in length

From $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ R is positive and decreasing in length

Together, the polar curve so far is this loop:

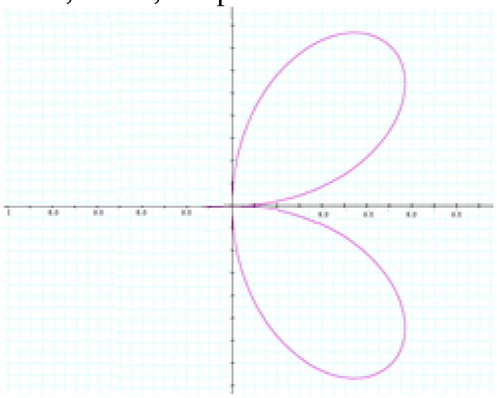


From $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{4}$ R is negative and increasing in length

But remember, -R = R + π, so we'll actually be drawing in the fourth quadrant instead of the second quadrant

From $\theta = \frac{3\pi}{4}$ to $\theta = \pi$ R is negative and decreasing in length, again in the fourth quadrant

Now, so far, the polar curve is:



From $\theta = \pi$ to $\theta = \frac{5\pi}{4}$ R is positive and increasing in length

From $\theta = \frac{5\pi}{4}$ to $\theta = \frac{3\pi}{2}$ R is positive and decreasing in length

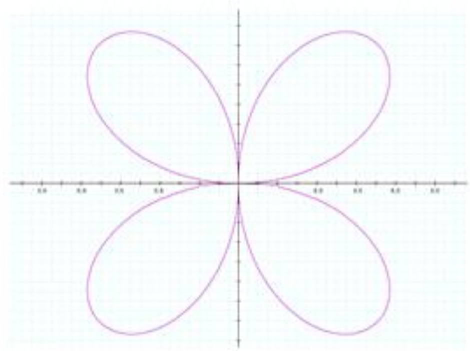
These give us a loop in the third quadrant, the quadrant swept out by those angles

From $\theta = \frac{3\pi}{2}$ to $\theta = \frac{7\pi}{4}$ R is negative and increasing in length

We are sweeping out angles in the fourth quadrant, but the negative R has us drawing in the second quadrant

From $\theta = \frac{7\pi}{4}$ to $\theta = 2\pi$ R is negative and decreasing in length, completing the loop in the second quadrant

We could continue graphing values from 2π onward, but they would simply retrace the completed shape already drawn:



As there is a value to being able to translate points in space between polar and Cartesian coordinates (described above), there is also value in being able to translate polar functions to Cartesian functions and vice-versa.

For example, take the Cartesian equation: y = x

We would like to convert this into a polar equation using only the variables R and θ. We do that using the same relationships we use for coordinates.

y = x becomes

R·sinθ = R·cosθ   or just   tanθ = 1

For an example of the reverse, consider the polar equation: R = 4·secθ

R = 4 · $\frac{1}{sin\theta}$ where cosθ = $\frac{x}{R}$

R = $\frac{4R}{x}$ so x = 4

As we did with parametric equations, we would like methods for finding tangents, arc lengths, and areas for polar curves.

Again noting that x = R·cosθ and y = R·sinθ

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dR}{d\theta}sin\theta+Rcos\theta}{\frac{dR}{d\theta}cos\theta-Rsin\theta}$ using the product rule

Let’s try this using our four-leaf clover of R = sin(2θ)

$\frac{dR}{d\theta}$ = 2cos(2θ) so $\frac{dy}{dx} = \frac{2\cos(2\theta)\cdot sin\theta + R^2 sin\theta\cdot cos\theta}{2\cos(2\theta)\cdot cos\theta - R^2 sin^2\theta}$

We should check this result at various angles:

θ = 0 rad, $\frac{dy}{dx} = \frac{0}{2} = 0$

θ = $\frac{\pi}{4}$ rad, $\frac{dy}{dx} = \frac{\frac{R^2}{2}}{\frac{-R^2}{2}} = -1$

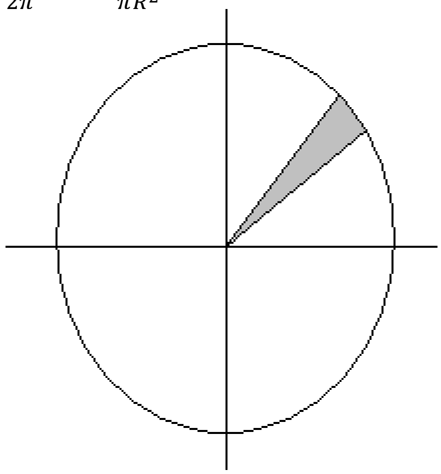
θ = $\frac{\pi}{2}$ rad, $\frac{dy}{dx} = \frac{2}{R^2} = \infty$ because R = 0 at θ = $\frac{\pi}{2}$ rad

[Recall R = sin(2θ) or look at the diagram]

We could continue this process looking at other angles, but they should also agree with the diagram.

Suppose we had some polar function, R = f(θ) and we wanted to find the area between two angles. The simplest case would be a circle. Suppose the radius of the circle is R and the area swept out is Δθ. We could use the ratio:

$\frac{\Delta\theta}{2\pi} = \frac{shaded\ area}{\pi R^2}$ or shaded area = $\frac{1}{2} \cdot R^2 \cdot \Delta\theta$



In differential notation, this becomes $dA = \frac{1}{2} \cdot R^2 \cdot d\theta$ where, again, R is some function of θ.

The area under a polar curve is then, $A = \int \frac{1}{2} \cdot R^2 \cdot d\theta$

What does this make the area of one leaf of our R = sin(2θ) four-leaf clover?

Using the equation above, we have $A = \int \frac{1}{2} \cdot \sin^2(2\theta) \cdot d\theta$

Let u = 2θ so that $d\theta = \frac{1}{2} du$ and also $\sin^2(u) = \frac{1 - \cos(2u)}{2}$

$$A = \int \frac{1}{2} \cdot \left[\frac{1 - \cos(2u)}{2}\right] \cdot \frac{1}{2}\ du = \frac{1}{8} \int [1 - \cos(2u)]du$$

$$A = \frac{1}{8} [\ u - \frac{1}{2}\sin(2u)\ ] = \frac{1}{8} [\ 2\theta - \frac{1}{2}\sin(4\theta)\ ] = \frac{1}{4}\theta - \frac{1}{16}\sin(4\theta)$$ with boundaries of 0 and $\frac{\pi}{2}$

$A = \frac{\pi}{8}$ for one leaf, which means $A = \frac{\pi}{2}$ for the whole curve

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Lastly is arc length.

As was shown earlier, $\frac{dy}{d\theta} = \frac{dR}{d\theta} sin\theta + Rcos\theta$ and $\frac{dx}{d\theta} = \frac{dR}{d\theta} cos\theta - Rsin\theta$

$(\frac{dy}{d\theta})^2 + (\frac{dx}{d\theta})^2$ then becomes:

$(\frac{dR}{d\theta})^2 \cdot \sin^2\theta + 2\cdot\frac{dR}{d\theta}\cdot\sin\theta\cdot\cos\theta + R^2 \cos^2\theta + (\frac{dR}{d\theta})^2 \cdot\cos^2\theta - 2\cdot\frac{dR}{d\theta}\cdot\sin\theta\cdot\cos\theta + R^2 \sin^2\theta$

which reduces to $(\frac{dR}{d\theta})^2 + R^2$

We saw with parametric equations that:

Arc length = $\int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt$ where t could be any parameter

If we take θ as our parameter so that t = θ, then

Arc length = $\int \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} \cdot d\theta$

This produces our final result of:

Arc length = $\int \sqrt{R^2 + (\frac{dR}{d\theta})^2} \cdot d\theta$

Applying this to our four-leaf clover leads to an overly complex integral, so let's just apply it to

$R = \cos^2(\frac{\theta}{2})$

Arc length = $\int \sqrt{R^2 + (\frac{dR}{d\theta})^2} \cdot d\theta = \int \sqrt{cos^4(\frac{\theta}{2}) + cos^2(\frac{\theta}{2})sin^2(\frac{\theta}{2})]} \cdot d\theta = \int \cos(\frac{\theta}{2}) \cdot d\theta$

Arc length = $2\cdot\sin(\frac{\theta}{2})$