Dot Product and Cross Product

The dot product (or scalar product) is the product of two vectors such that:

$\mathbf{A} \bullet \mathbf{B} = (A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z)$

It has the following properties:

1. $\mathbf{A} \bullet \mathbf{A} = |A|^2$ where |A| is the magnitude of **A**

2. $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

3. $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

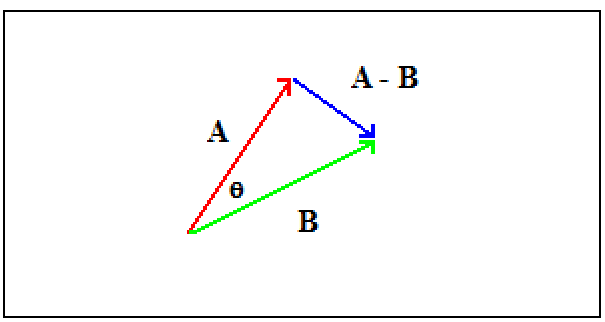
4. $(c\mathbf{A}) \bullet \mathbf{B} = c(\mathbf{A} \bullet \mathbf{B})$

5. $0 \bullet \mathbf{A} = 0$

As before, these are all easy to prove by writing **A** and **B** in their component forms.

Another commonly used form of the dot product is:

$\mathbf{A} \bullet \mathbf{B} = |A| \cdot |B| \cdot \cos\theta$ or $\cos\theta = \frac{\mathbf{A} \bullet \mathbf{B}}{|A| \cdot |B|}$ where θ is the angle between **A** and **B**



We can see this using the diagram above and the law of cosines.

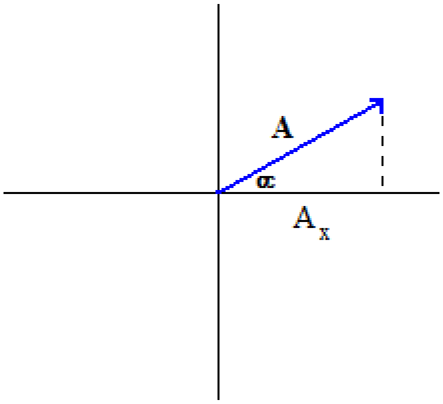
$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}| \cdot |\mathbf{B}| \cdot \cos\theta$

$(\mathbf{A} - \mathbf{B}) \bullet (\mathbf{A} - \mathbf{B}) = \mathbf{A} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B} - 2|\mathbf{A}| \cdot |\mathbf{B}| \cdot \cos\theta$

$\mathbf{A} \bullet \mathbf{A} - 2\mathbf{A} \bullet \mathbf{B} + \mathbf{B} \bullet \mathbf{B} = \mathbf{A} \bullet \mathbf{A} + \mathbf{B} \bullet \mathbf{B} - 2|\mathbf{A}| \cdot |\mathbf{B}| \cdot \cos\theta$

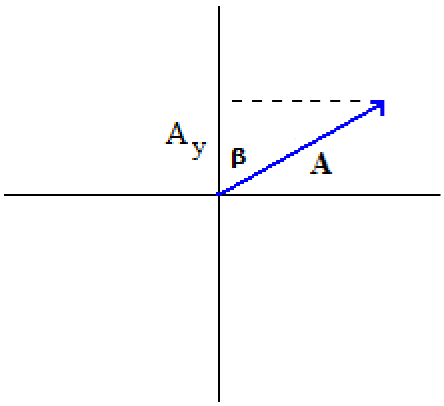
$\cos\theta = \frac{\mathbf{A} \bullet \mathbf{B}}{|A| \cdot |B|}$

From $\mathbf{A} \bullet \mathbf{B} = |A| \cdot |B| \cdot \cos\theta$, we can see that if **A** and **B** are perpendicular, the dot product is zero, if parallel, the dot product is $|A| \cdot |B|$, and if antiparallel, the dot product is $-|A| \cdot |B|$.



It is common to diagram a vector **A** with an angle off of the positive x-axis and to compute the x-component of that vector as A_x = |**A**|cosα. This would be true even if **A** were not in the xy plane.

But to avoid being provincial, it is equally true that the y-component of the vector can be computed with A_y = |**A**|cosβ where β is the angle off of the positive y-axis.



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Likewise, A_z = |**A**|cosγ where γ is the angle between **A** and the positive z-axis.

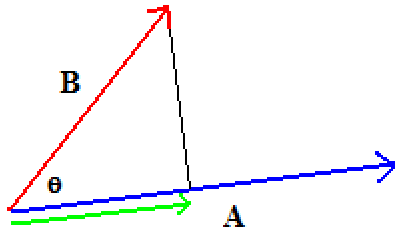
cosα, cosβ, and cosγ are called directional cosines where

$\cos\alpha = \frac{A_x}{|\mathbf{A}|}$ $\cos\beta = \frac{A_y}{|\mathbf{A}|}$ $\cos\gamma = \frac{A_z}{|\mathbf{A}|}$

We can then write **A** = (A_x, A_y, A_z) = |**A**|·(cosα, cosβ, cosγ)

or $\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$ = (cosα, cosβ, cosγ) where $\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$ is the unit vector of **A**

The angles α, β, and γ are known as the *directional angles*.



Suppose we have two vectors **A** and **B**. We can draw a line from the tip of **B** to **A** so that this line is perpendicular to **A**. This creates the green vector which we call the projection of vector **B** onto vector **A** or $\text{proj}_{\mathbf{A}}\mathbf{B}$.

The signed magnitude (positive if **A** and the projection are parallel, negative if they are antiparallel) is called the component of **B** along **A** or $\text{comp}_{\mathbf{A}}\mathbf{B}$.

Because $\text{comp}_{\mathbf{A}}\mathbf{B} = |\mathbf{B}|\cos\theta$, we can say that the dot product of **A** and **B** is the length of **A** times the component of **B** along **A**.

$\mathbf{A}\bullet\mathbf{B} = |A|\cdot|B|\cdot\cos\theta$

Rearranged, $|B|\cdot\cos\theta = \frac{\boldsymbol{A}}{|A|} \bullet \boldsymbol{B}$

The scalar component of **B** along **A** equals the unit vector of **A** dot **B.** Conceptually, you can think of this as "how much of **B** is parallel to **A**".

To turn this back into a vector (the projection rather than the component), we multiply by the unit vector for **A**, which is $\frac{\boldsymbol{A}}{|A|}$.

$\text{proj}_{\mathbf{A}}\mathbf{B} = \frac{\boldsymbol{A}}{|A|} \bullet \boldsymbol{B} \cdot \frac{\boldsymbol{A}}{|A|} = \frac{\boldsymbol{A}\bullet\boldsymbol{B}}{|A|^2} \cdot \boldsymbol{A}$

Conceptually, this is still "how much of **B** is parallel to **A**", though now we have pointed it in the direction of **A**.

We can think of the dot product as a quantity which signifies how parallel two vectors are. We also find it useful to have a measurement of how perpendicular two vectors are. This is what the cross product (or vector product) produces.

Given two vectors:
$\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$

the cross product, **A x B** is a vector perpendicular to both **A** and **B** and has components:

$\mathbf{A \times B} = (A_y\cdot B_z - A_y\cdot B_z, A_z\cdot B_x - A_x\cdot B_z, A_x\cdot B_y - A_y\cdot B_x)$

It is, however, much easier to remember when written as a determinant:

$$\mathbf{A \times B} = \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boldsymbol{A_x} & \boldsymbol{A_y} & \boldsymbol{A_z} \\ \boldsymbol{B_x} & \boldsymbol{B_y} & \boldsymbol{B_z} \end{array}$$

We can show that **A x B** is perpendicular to both **A** and **B** by showing (**A x B**)•**A** = 0 and (**A x B**)•**B** = 0

For example:

$(\mathbf{A \times B})\bullet\mathbf{A} = (A_y \cdot B_z - A_y \cdot B_z,\ A_z \cdot B_x - A_x \cdot B_z,\ A_x \cdot B_y - A_y \cdot B_x)\bullet(A_x, A_y, A_z) =$

$(A_y \cdot B_z \cdot A_z - A_y \cdot B_z \cdot A_z,\ A_z \cdot B_x \cdot A_y - A_x \cdot B_z \cdot A_y,\ A_x \cdot B_y \cdot A_z - A_y \cdot B_x \cdot A_z) = (0, 0, 0)$

The second way of writing the cross product is:

$\mathbf{A \times B} = |A| \cdot |B| \cdot \sin\theta$ where θ is the smaller angle between vectors **A** and **B**

We can show the equivalence of these two ways of determining the cross product as follows:

$|\mathbf{A \times B}|^2 = (A_y \cdot B_z - A_y \cdot B_z)^2 + (A_z \cdot B_x - A_x \cdot B_z)^2 + (A_x \cdot B_y - A_y \cdot B_x)^2$

because of the Pythagorean theorem

Expanded and then rearranged, this becomes:

$(A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2$

$= |A|^2 \cdot |B|^2 - (\mathbf{A} \bullet \mathbf{B})^2$

$= |A|^2 \cdot |B|^2 - |A|^2 \cdot |B|^2 \cdot \cos\theta$

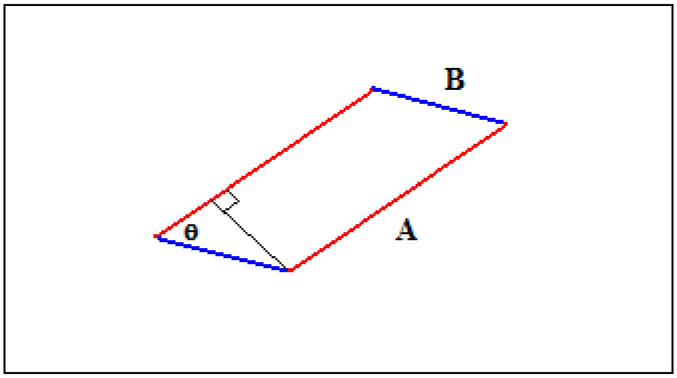
$= |A|^2 \cdot |B|^2 \cdot (1 - \cos^2\theta)$

$= |A|^2 \cdot |B|^2 \cdot \sin^2\theta$

Therefore, $|\mathbf{A \times B}| = |\mathbf{A}| \cdot |\mathbf{B}| \cdot \sin\theta$

This form tells us that if the two vectors are parallel or antiparallel, **A x B** = 0

However, this form does not indicate the direction of the cross-product. For this, we can use the right-hand rule. Orient the palm of the right hand in the direction of the first vector (**A**) and then sweep the fingers to align with the second vector (**B**). The direction of the thumb indicates the direction of the cross product.



We can also see in the diagram above that |**A x B**| = |**A**|·|**B**|·sinθ and is the area of a parallelogram with sides |**A**| and |**B**|.

As before, the following list of cross product properties can easily be proved by writing the vectors in component form and completing the arithmetic:

1. **A x B** = - (**B x A**)

2. (c**A**) **x B** = c(**A x B**) = **A x** (c**B**)

3. **A x** (**B** + **C**) = **A x B** + **A x C**

4. (**A** + **B**) **x C** = **A x C** + **B x C**

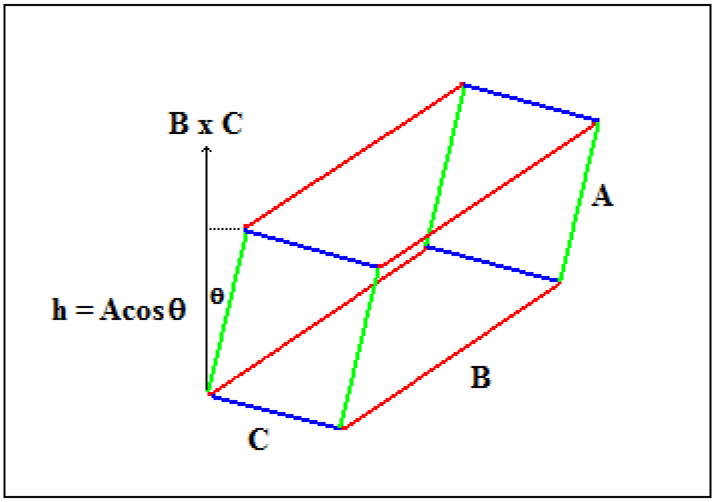
5. **A•**(**B x C**) = (**A x B**)**•C**

6. **A x** (**B x C**) = (**A•C**)·**B** – (**A•B**)·**C**

Notice from (1) that the cross product is not commutative and from (6) that the cross product is not associative. The left side of the equation in (5) is called the scalar triple product while the left side of the equation in (6) is called the vector triple product.

Expanding the scalar triple product into its components, you will find it can be written as the determinant:

$$\mathbf{A}\bullet(\mathbf{B}\ \mathbf{x}\ \mathbf{C}) = \begin{matrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{matrix}$$



This too has a geometric meaning seen in the diagram above, a parallelepiped with sides A, B, and C. The parallelepiped has a base with an area |**B x C**| and a height of |A|cosθ where θ is the angle between **A** and the vector **BxC**. So the volume of the parallelepiped = |A|·|(**B x C**)|·cosθ or

V = **A•(BxC)**