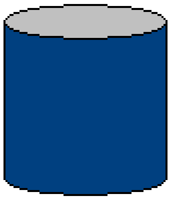
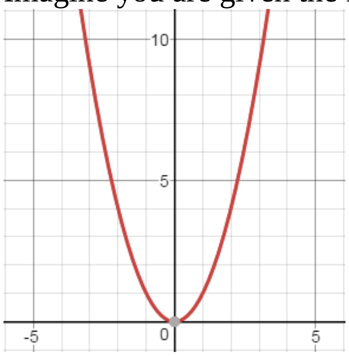
# Equations of Cylinders and Quadratic Surfaces

We think of the term *cylinder* to mean the shape of a tin can:

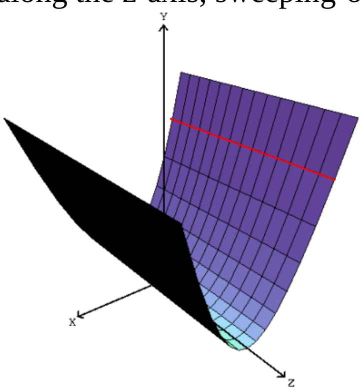


but, in mathematics, it has a more general definition.

Imagine you are given the function, $y = x^2$. This is easy enough to graph in the x-y plane:

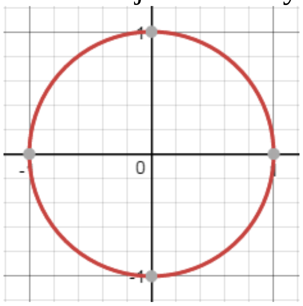



To imagine this as a cylinder, just recognize that the function $y = x^2$ makes no reference to z, so it is true for all values of z. This means the parabola above can be extended back and forward along the z-axis, sweeping-out what is called a *parabolic cylinder*.

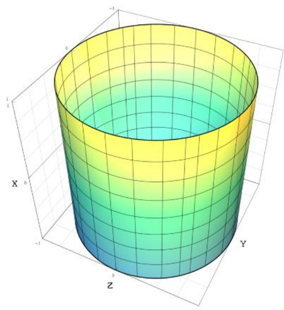



The red line in the picture above is called a *ruling*, a line parallel to the unreferenced z-axis that passes through the defined plane curve of $y = x^2$.

We could just as easily take the plane curve, $y^2 + z^2 = 1$




and imagine sweeping it back and forth along the x-axis to create the cylinder:

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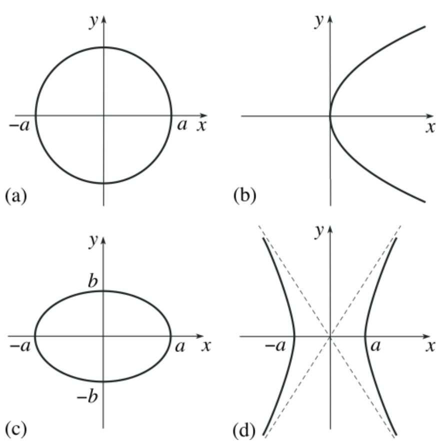
Here are the four standard conic sections:




which can all be written in the general form:

$Ax^2 + By^2 + Cx + Dy + E = 0$

We can extend these shapes into three dimensions by introducing the z-axis and using the general form:

$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

However, by rotating or shifting the Cartesian axes, we can use two simpler, standard forms:

(1) $Ax^2 + By^2 + Cz^2 + J = 0$ or (2) $Ax^2 + By^2 + Iz = 0$

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Let's take the second equation and write it in the following form (where a, b, and c are positive constants):

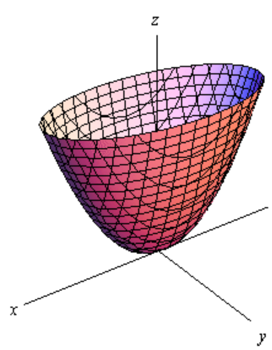
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ recognizing that there are two possibilities: if both left-hand terms have the same sign or if the two left-hand terms have opposite signs

If the signs are the same, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$. To determine what this looks like, chose a certain value for z and then visualize what that curve (called a *trace*) will look like parallel to the x-y plane.

For example, when z = 0, both x = 0 and y = 0, which gives us a point at the origin of the Cartesian axes.

When z = 1, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{c}$ which is the curve of an ellipse parallel to the x-y plane at a height of z = 1.

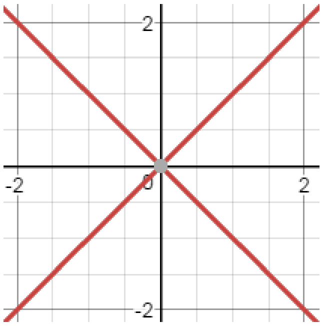
When z = 2, we have a similar ellipse at a height of z = 2 but with a larger circumference. As z increases, so does this circumference, suggesting a shape known as an *elliptic paraboloid*:



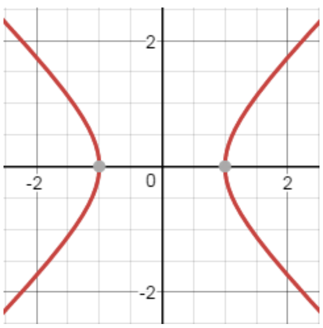

We can see that the axis of the shape is the linear term in the equation and also that the equation becomes parabolic if we set x or y equal to some constant (as agrees with the diagram).

If the signs are instead different, we have $\frac{x^2}{a^2}$ - $\frac{y^2}{b^2}$ = $\frac{z}{c}$. Again, we choose various values for z and imagine what the curve looks like at that height parallel to the x-y plane.

When z = 0, we have y = ±$\sqrt{\frac{b}{a}}$·x, which is a simple x-shape in the x-y plane:



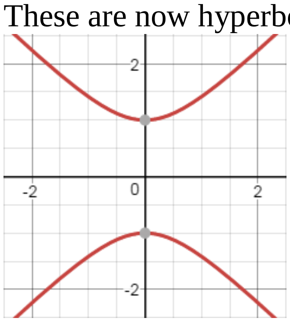

When z = 1, we have $\frac{x^2}{a^2}$ - $\frac{y^2}{b^2}$ = $\frac{1}{c}$ which forms a hyperbola with vertices along the x-axis.




As z increases positively, we continue to have hyperbolas with vertices along the x-axis as these vertices move further and further from the origin.

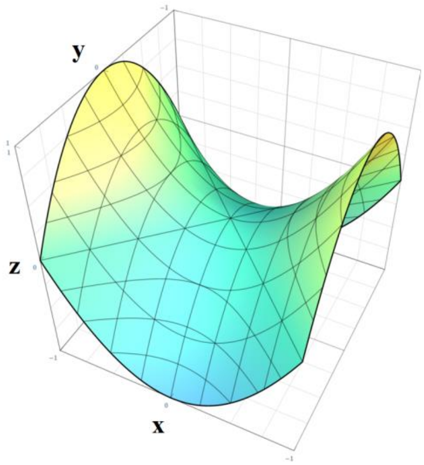
When z = -1, we now have $\frac{x^2}{a^2}$ - $\frac{y^2}{b^2}$ = $\frac{-1}{c}$ or $\frac{y^2}{b^2}$ - $\frac{x^2}{a^2}$ - = $\frac{1}{c}$

These are now hyperbolas with vertices along the y-axis.




And as z becomes increasingly negative, we continue to have hyperbolas with vertices along the y-axis which have these vertices moving further and further from the origin.

Altogether, these traces give use what is known as a *hyperbolic paraboloid*:




a common example of which is the Pringle potato chip.

Again, we can look at $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ and see that if we set x or y equal to zero, we get the parabolic trace within the surface.

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Now, let's take the equation $Ax^2 + By^2 + Cz^2 + J = 0$

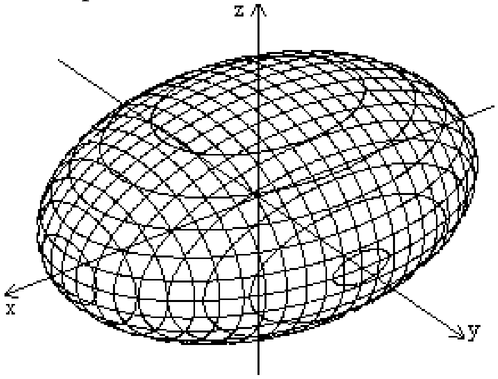
We can rewrite this as $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$ where a, b, c, and k are constants. We have two possibilities:

(1) all three left-hand terms have the same sign
(2) one left-hand term has a sign opposite the other two

In the first case, the only real possibility is if a, b, c, and k are all positive. If all are negative, we just multiply the equation by -1.

If we take the case of z = 0, we have an ellipse in the x-y plane. As z increases positively or negatively, we still have elliptical curves parallel to the x-y plane, but their circumference must be decreasing as the $z^2$ term gradually takes more and more of k's total value. Eventually, we reach a point above the x-y plane where $\frac{z^2}{c^2} = k$ and where x and y equal zero along the z-axis. There is a similar point below the x-y plane.

By the symmetry of the equation, we can equally imagine these elliptical traces shrinking in size as we move away from the origin along either the x or y axis, giving us an overall shape called an ellipsoid:

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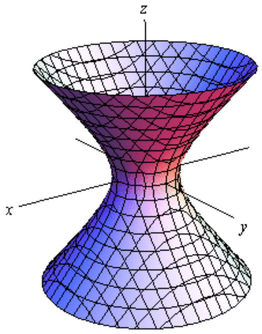
Now suppose we take the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$ and make the third term negative so that:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{k} + \frac{z^2}{c^2}$

When z = 0, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$ which is an ellipse in the x-y plane.

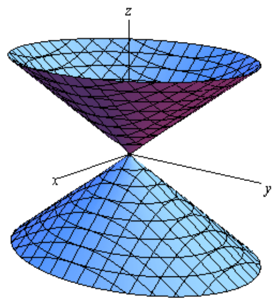
As z becomes increasingly positive or negative, we still have the equation of an ellipse parallel to the x-y plane, but the circumference of the ellipse increases with z.

Altogether, these traces suggest the shape known as the *hyperboloid of one sheet*:

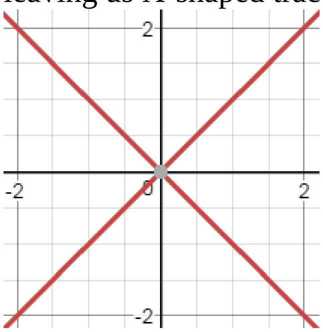



Looking at $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k$, we see that if we set x or y equal to zero, we get the equation of a hyperbola, which is true of those traces parallel to the x-z or y-z plane.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = k$ and k = 0, this simplifies to a *cone*.

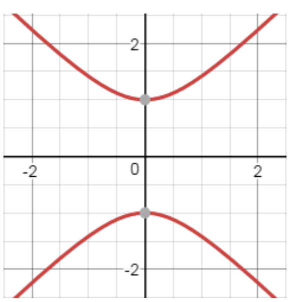



When z = 0, x and y must also equal zero. If we set x or y equal to zero, we have y ∝ z or x ∝ z, leaving us X-shaped traces in the the x-z or y-z plane.




If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ for the cone and we set y equal to +1 or -1, we get the equation for the hyperbola that is a trace of the cone for a plane parallel to the x-z plane.

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = \frac{1^2}{b^2}$$




As these y-values become greater, the vertices of the hyperbolas move further and further from the y-axis. And by the symmetry of the cone, we can make a similar argument for values of x.

What if we return to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$ and make *two* of the left-hand terms negative, so that:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = k$$

Setting k equal to zero is nothing new; this is just the equation of the cone again, but now along the x-axis if we multiply all three terms by -1.

If k is instead some positive constant, we see that x can never be zero, so nothing exists in the y-z plane.

If we set y equal to zero, we have a trace hyperbola in the x-z plane. As y increases, we can imagine swinging that term over to the right-hand side of the equation, adding to *k* and moving the vertices of these trace hyperbolas away from the y-z plane.

The above paragraph applies equally well for z-values, suggesting a shape known as a *hyperboloid of two sheets*.