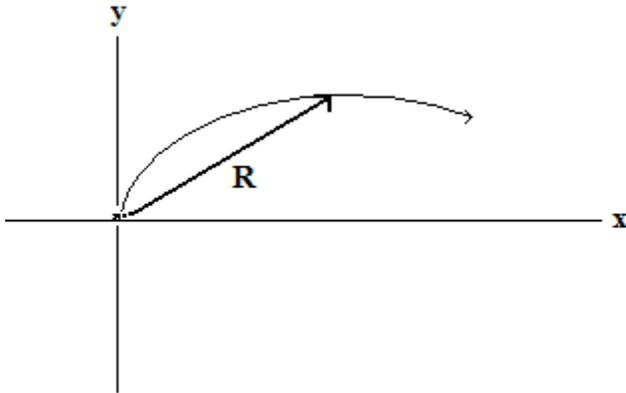


Vector Functions



Let's return to the example of a cricket jumping off the ground with an initial velocity v_0 at an angle θ .

We can write the position of the cricket as a *vector function*, a function which produces a vector in the range for every time in the domain. Here, that function would be:

$$\mathbf{R} = \left\langle v \cdot (\cos\theta) \cdot t, v \cdot (\sin\theta) \cdot t - \frac{g \cdot t^2}{2}, 0 \right\rangle$$

or, in component form:

$$\mathbf{x} = v \cdot (\cos\theta) \cdot t$$

$$\mathbf{y} = v \cdot (\sin\theta) \cdot t - \frac{g \cdot t^2}{2}$$

$$\mathbf{z} = 0$$

The set of all points at positions \mathbf{R} make up the path of the cricket's motion, known as the *space curve*.

In the same way we can differentiate real value functions, we can differentiate vector functions, which is really nothing more than differentiating the component functions.

$$\text{If } \mathbf{R} = \left\langle v \cdot (\cos\theta) \cdot t, v \cdot (\sin\theta) \cdot t - \frac{g \cdot t^2}{2}, 0 \right\rangle$$

$$\text{then } \mathbf{v} = \left\langle v \cdot (\cos\theta), v \cdot (\sin\theta) - g \cdot t, 0 \right\rangle$$

$$\text{and } \mathbf{a} = \left\langle 0, -g, 0 \right\rangle$$

The rules for differentiating vector functions are essentially the same as those for real value functions, with some additions for vector multiplication:

$$1. \frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$2. \frac{d}{dt}[c \cdot \mathbf{u}(t)] = c \cdot \mathbf{u}'(t)$$

$$3. \frac{d}{dt}[f(t) \cdot \mathbf{u}(t)] = f(t)' \cdot \mathbf{u}(t) + f(t) \cdot \mathbf{u}'(t)$$

$$4. \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t)' \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad \text{for dot products}$$

$$5. \frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t)' \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad \text{for cross products}$$

$$6. \frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \quad \text{chain rule}$$

These are not too difficult to prove if the vector functions are expanded into component form.

We can likewise integrate vector functions. For example, if we know $\mathbf{R}_0 = (1, 2, 3)$ and $\mathbf{v} = \langle 2t, 5, 6t^2 \rangle$, then

$$\mathbf{R} = \langle t^2 + 1, 5t + 2, 2t^3 + 3 \rangle$$

Often, the space curves that arise in these vector functions are much more complex than the simple parabola of the cricket's jump. For example, we may need to visualize the curve of:

$$\mathbf{R} = \langle t, \sin(t), t \rangle$$

or, in parametric form:

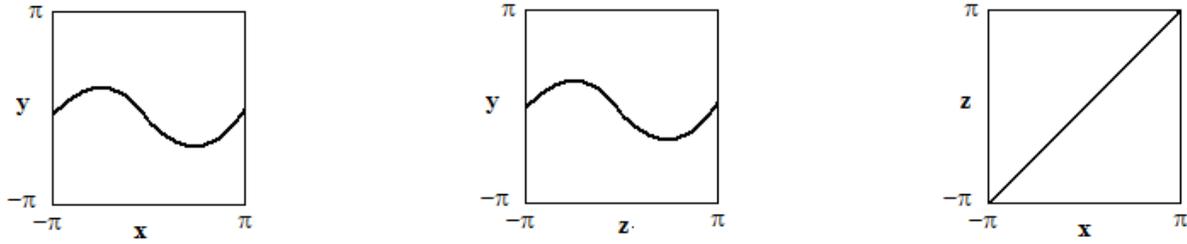
$$x = t$$

$$y = \sin(t)$$

$$z = t$$

Let's imagine this vector function existing in a cube with sides of 2π where the center of the box has coordinates $(0, 0, 0)$. One method that can be helpful is to imagine what the cube will look like from various perspectives.

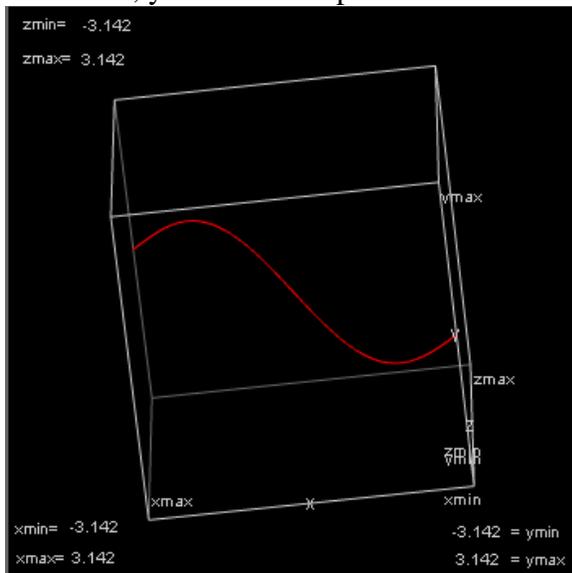
Rearranging the above equations, we can see that $x = z$, $y = \sin(x)$ and $y = \sin(z)$. Those graphs (or projections) would then look like this:



If it is still difficult to visualize the three-dimensional curve, you can select a series of values for t and consider where points exist within our cube by using the time-function equations.

For example, at $t = -\pi$, we would have a point at the center of the back-left edge. At time $t = 0$, we would have a point at the center of the box. At time $t = \pi$, we would have a point at the center of the front-right edge.

In the end, you have a shape that looks like this:



The above was drawn with a 3D parametric graphing program at:

<http://www.math.uri.edu/~bkaskosz/flashmo/parcur/>

Graphing utilities like this are very helpful in visualizing complex curves. Rotating the cube to see various perspectives confirms the three projections above.

Another way of understanding this curve is to realize that if $x = z$ for all points, then the curve must lie on the $x=z$ plane that cuts the cube in half from the back-left edge to the right-front edge.

This can also be thought-of as the intersection between the $x=z$ plane and the $y = \sin(x)$ cylinder or, equally well, the intersection between the $x=y$ plane and the $y = \sin(z)$ cylinder.

Lastly, it is useful to find, for any spatial curve, the vector function for the tangent line at any point along the curve. This can be done simply by differentiation. For example, if:

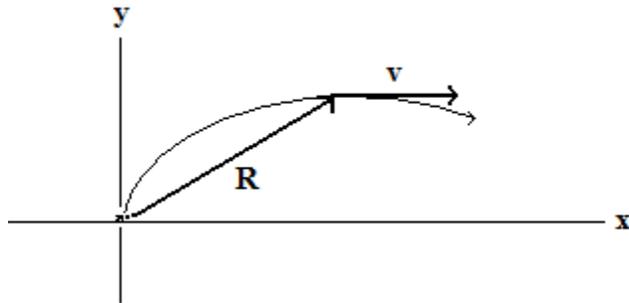
$$\mathbf{R} = \langle t, t^2, t^3 \rangle$$

then the tangent vector, $\mathbf{R}' = \langle 1, 2t, 3t^2 \rangle$

If we only want the unit tangent vector (which has a magnitude of one), we divide by the magnitude of the derivative:

$$\text{unit } \mathbf{T} = \frac{\mathbf{R}'}{|\mathbf{R}'|} = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1^2 + (2t)^2 + (3t^2)^2}}$$

This is really nothing new. When we took the cricket's position function and differentiated to find the velocity function, that was also finding the tangent function.



These tangent lines provide the instantaneous direction of motion as you move along the spatial curve.