# Arc Length, Curvature, and Equations of Motion

In the first set of notes, we saw the equation for arc length in two-dimensions as:

Arc length = $\int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt$

Using the same Pythagorean theorem, we can extend this simply into three-dimensions:

Arc length = $\int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \cdot dt$

or, more compactly, arc length = $\int |\boldsymbol{R}'(t)| \cdot dt$

Conceptually, this states that the distance travelled along a curve equals the sum of little steps, where each little step is the instantaneous speed multiplied by a little span of time.

This is because $\mathbf{R}(t) = x(t)\cdot\mathbf{i} + y(t)\cdot\mathbf{j} + z(t)\cdot\mathbf{k}$

therefore, $\mathbf{R}'(t) = x'(t)\cdot\mathbf{i} + y'(t)\cdot\mathbf{j} + z'(t)\cdot\mathbf{k}$ which are three vectors all perpendicular, so

$|\mathbf{R}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$

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At times it is helpful, in finding arc lengths, to use the process of *parameterization*.

For instance, suppose we have the function: $\mathbf{R}(t) = (\cos t)\cdot\mathbf{i} + (\sin t)\cdot\mathbf{j} + (t)\cdot\mathbf{k}$

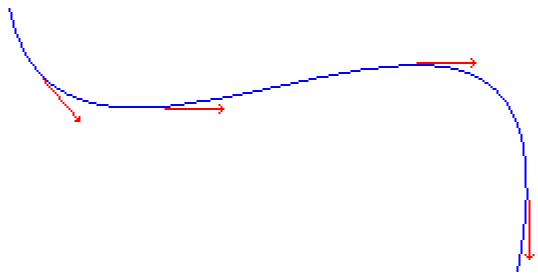
Using this, we could find the position along the curve at any given, t, just by inserting the time into that equation. But suppose we want the position along the curve after having traveled a certain arc length, s, from an initial position of (1, 0, 0). What we want is **R**(s), or more precisely, **R**(s(t)). We want to change the parameter from *t* to *s*.

Using arc length = $\int |\boldsymbol{R}'(t)| \cdot dt$ we first find

$|\mathbf{R}'(t)| = \sqrt{[-\sin(t)]^2 + [\cos(t)]^2 + [1]^2} = \sqrt{2}$ which is the speed

Then arc length, $s = \int_0^t \sqrt{2}\, dt = \sqrt{2}\cdot t$

Therefore, $t = \frac{s}{\sqrt{2}}$ and $\mathbf{R}(t) = (\cos \frac{s}{\sqrt{2}})\cdot\mathbf{i} + (\sin \frac{s}{\sqrt{2}})\cdot\mathbf{j} + (\frac{s}{\sqrt{2}})\cdot\mathbf{k}$



Recall that we defined the unit tangent vector as

$$\mathbf{T} = \frac{\boldsymbol{R'}}{|\boldsymbol{R'}|}$$ which can be thought of as velocity divided by speed

and which indicates the direction of the curve at any point along the curve. A smooth curve is one in which **T** changes continuously, in other words, one in which there are no sharp corners where **T** would be undefined.

We would like a concept to represent how quickly the unit tangent vector changes as we take steps of ds along the curve, and this is provided by the concept of *curvature*, κ.

$$\kappa = \left| \frac{d\boldsymbol{T}}{ds} \right|$$

However, most curves are given to us as functions of the parameter, t, so we can write:

$$\frac{d\boldsymbol{T}}{dt} = \frac{d\boldsymbol{T}}{ds} \cdot \frac{ds}{dt}$$ so that $$\kappa = \left| \frac{d\boldsymbol{T}}{ds} \right| = \left| \frac{d\boldsymbol{T} / dt}{ds / dt} \right|$$ where $\frac{ds}{dt} = |\mathbf{R}'(t)|$

so

$$\kappa = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{R}'(t)|}$$

For a particle moving along a curve, we can think of the denominator of this equation as the speed of the particle. But the curvature of a line is independent of the speed along which it is traveled, so the above equation tells us that if we double the speed at which a particle moves along a curve, we will double the rate at which the unit tangent vector changes. This should match your intuition.

Another useful equation for curvature is:

$$\kappa = \frac{|\boldsymbol{R}' \times \boldsymbol{R}''|}{|\boldsymbol{R}'|^3}$$

the derivation of which is not particularly instructive, so I will leave it for an appendix at the end.

If the curve is restricted to two-dimensions, it will take the form of something like y = f(x) so that

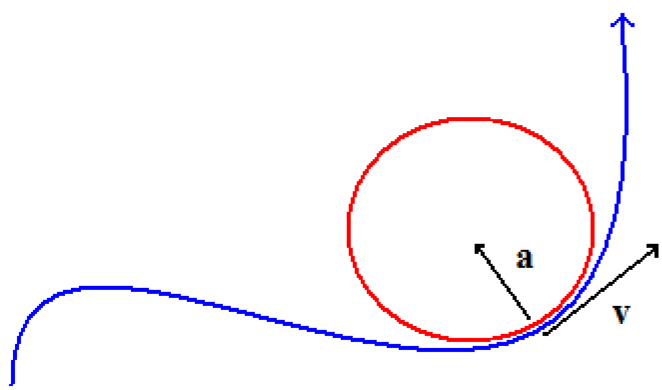
$\mathbf{R}(x) = x\cdot\mathbf{i} + f(x)\cdot\mathbf{j}$
$\mathbf{R}'(x) = \mathbf{i} + f'(x)\cdot\mathbf{j}$
$\mathbf{R}''(x) = f''(x)\cdot\mathbf{j}$

$|\boldsymbol{R}' \times \boldsymbol{R}''|$ is then $|\,[\mathbf{i} + f'(x)\cdot\mathbf{j}]\ \mathbf{x}\ [f''(x)\cdot\mathbf{j}]\,| = |f''(x)\cdot\mathbf{k}| = f''(x)$

and

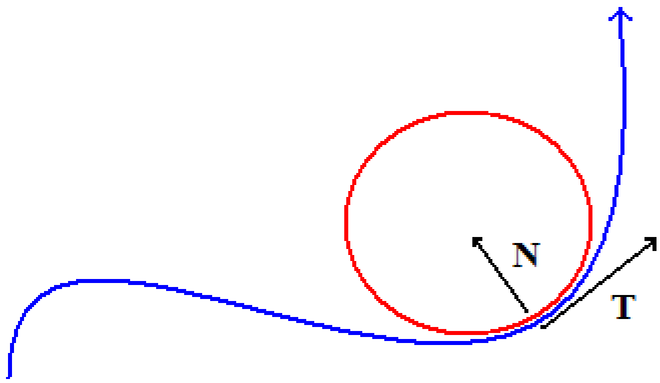
$|\boldsymbol{R}'|^3 = [1 + (f'(x))^2]^{3/2}$ because $|\mathbf{R}'| = \sqrt{[\mathbf{i} + f'(x)\cdot\mathbf{j}\,]^2} = [1 + (f'(x))^2]^{1/2}$

$\kappa = \dfrac{|\boldsymbol{R}' \times \boldsymbol{R}''|}{|\boldsymbol{R}'|^3}$ is then also $\kappa = \dfrac{f''(x)}{[1+(f'(x))^2]^{1/2}}$




Take the blue curve above. Any point along the curve can be considered a point where the curve is in contact with a circle of equivalent curvature. The acceleration is centripetal, so the acceleration vector is orthogonal, or perpendicular, to the velocity vector and **a** = **v**'(t). If we want a unit acceleration vector, we just divide by the magnitude of the acceleration so that:

unit **a** = $\dfrac{\boldsymbol{v}'(t)}{|\boldsymbol{v}'(t)|}$



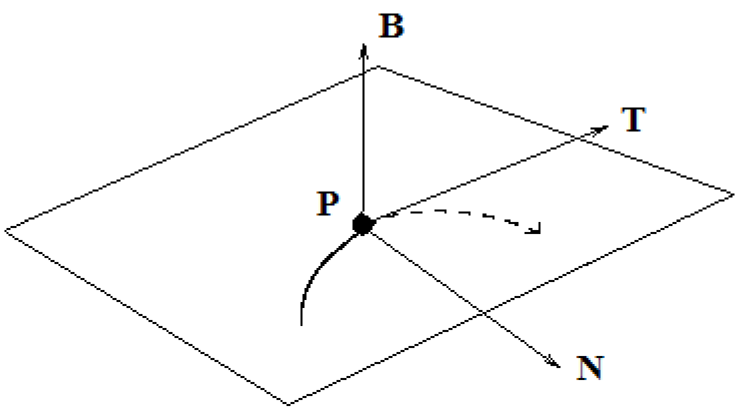

For curves in general, we let **T** represent the unit tangent vector and **N** represent the unit normal vector such that:

$$\mathbf{N} = \frac{\boldsymbol{T'}(t)}{|\boldsymbol{T'}(t)|}$$

One step further, we define the unit *binormal vector* as the cross-product:

**B**(t) = **T**(t) **x** **N**(t)

so that the unit tangent vector, unit normal vector, and unit binormal vector are all perpendicular to each other.




The plane containing the vectors **N** and **B** is called the *normal plane* at P because it is a plane normal to the unit tangent vector.

The plane containing the vectors **T** and **N** is called the *osculating plane* and is a plane that contains the **ds** passing through point P.

If the above diagram shows a curve entirely in the plane drawn, then the oscillating plane is the plane shown.  If we draw a circle in that plane so that P sits on the circumference, with the curve and circle having the same tangent line at P, this is called an *osculating circle* or *circle of curvature*. That was the red circle in the two previous diagrams.  The radius of this circle is the reciprocal of the curvature, or:

Radius of curvature, $\rho = \frac{1}{\kappa}$

If we think of the tangent vector as representing velocity, we can see this agrees with our definition of centripetal acceleration.

$$\kappa = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{R}'(t)|} = \frac{\frac{d}{dt}(\frac{\boldsymbol{v}}{|v|})}{|v|} = \frac{a}{v^2}$$

$$\kappa = \frac{1}{\rho} = \frac{1}{R} = \frac{a}{v^2} \quad \text{or} \quad a = \frac{v^2}{R}$$

And to review kinematics:

If **R**(t) represents a position vector then the velocity vector

**v**(t) = **R**'(t)

where speed is |**v**(t)| or |**R**'(t)|

and the acceleration vector

**a**(t) = **v**'(t) = **R**''(t)

Like any vector, we can decompose the acceleration vector into two components, one parallel to the velocity and one perpendicular to the velocity.

Begin with the idea that velocity is the speed of an object multiplied by the unit tangent vector:

$\mathbf{v} = v \cdot \mathbf{T}$

$\mathbf{a} = \mathbf{v}' = v \cdot \mathbf{T}' + v' \cdot \mathbf{T}$ by the product rule

The first term represents centripetal acceleration, changing the direction of motion at a given speed, v. The second term represents tangential acceleration, changing speed in a given direction, **T**.

$\kappa = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{R}'(t)|} = \frac{|\boldsymbol{T}'(t)|}{|v|}$ so $|\mathbf{T}'| = \kappa \cdot v$

Also, $\mathbf{N} = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|}$ so $\mathbf{T}' = |\mathbf{T}'| \cdot \mathbf{N} = \kappa \cdot v \cdot \mathbf{N}$

Therefore, $\mathbf{a} = \kappa \cdot v^2 \cdot \mathbf{N} + v' \cdot \mathbf{T}$

Again, the first term is centripetal, the second term is tangential.

At times, it is preferable to have these equations in terms of **R**, **R**', and **R**''.

First, we take the dot product of velocity and acceleration:

$\mathbf{v} \bullet \mathbf{a} = (v \cdot \mathbf{T}) \bullet (\kappa \cdot v^2 \cdot \mathbf{N} + v' \cdot \mathbf{T}) = v \cdot v'$

Tangential acceleration magnitude, $a_T = |\, v' \cdot \mathbf{T} \,| = v' = \frac{\mathbf{v} \bullet \mathbf{a}}{v} = \frac{\boldsymbol{R}'(t) \bullet \boldsymbol{R}''(t)}{|\boldsymbol{R}'(t)|}$

Centripetal or normal acceleration magnitude, $a_N = |\, \kappa \cdot v^2 \cdot \mathbf{N} \,| = \kappa \cdot v^2 = \frac{|\boldsymbol{R}' \times \boldsymbol{R}''|}{|\boldsymbol{R}'|^3} \cdot |\boldsymbol{R}'|^2 = \frac{|\boldsymbol{R}' \times \boldsymbol{R}''|}{|\boldsymbol{R}'|}$

Appendix: proof that $\kappa = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{R}'(t)|}$ produces $\kappa = \frac{|\boldsymbol{R}' \text{x } \boldsymbol{R}''|}{|\boldsymbol{R}'|^3}$

$\mathbf{T} = \frac{\boldsymbol{R}'}{|\boldsymbol{R}'|}$ and $|\mathbf{R}'| = \frac{ds}{dt}$ so $\mathbf{R}' = |\mathbf{R}'|\cdot\mathbf{T} = \frac{ds}{dt}\cdot\mathbf{T}$

Differentiating **R**' with the product rule: $\mathbf{R}'' = \frac{d^2s}{dt^2}\cdot\mathbf{T} + \frac{ds}{dt}\cdot\mathbf{T}'$

Then the cross product, $\mathbf{R}' \mathbf{\ x\ } \mathbf{R}'' = \frac{ds}{dt}\cdot\mathbf{T} \mathbf{\ x\ } (\frac{d^2s}{dt^2}\cdot\mathbf{T} + \frac{ds}{dt}\cdot\mathbf{T}') = (\frac{ds}{dt})^2\cdot(\mathbf{T} \mathbf{\ x\ } \mathbf{T}')$

**T** has a magnitude of one and **T** and **T**' are perpendicular, so

$|\ \mathbf{R}' \mathbf{\ x\ } \mathbf{R}''\ | = (\frac{ds}{dt})^2\cdot\ |\mathbf{T}'|$ or $|\mathbf{T}'| = \frac{|\boldsymbol{R}' \text{x } \boldsymbol{R}''|}{(ds/dt)^2}$

$$\kappa = \frac{|\boldsymbol{T}'|}{|\boldsymbol{R}'|} = \frac{|\boldsymbol{R}' \text{x } \boldsymbol{R}''|}{|\boldsymbol{R}'|^3}$$